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Dynamics of coupled multistable oscillators

by

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The project is focused on the following issues:

- IIdentification of possible synchronous responses of coupled oscillators, and existence of synchronous clusters as well
- Dynamical analysis of identical coupled systems suspended on elastic structure in context of the energy transfer between systems
- Investigation of phase or frequency synchronization effects in groups of coupled non-identical systems
- Developing methods of motion stability control of considered systems
- Investigation of time delay effects in analyzed systems
- Developing the idea of energy extraction from ocean waves using a series of rotating pendulums

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Contents

Chapter 1

Introduction

We live in a world of oscillations. Everything that surrounds us, from the engines of our cars and computer processors by passing butterfly flapping its wings, ending the contractions of our own hearts, have one thing in common - generates rhythm. These are not isolated objects. On the contrary, they co-exist in the environment and interact with each other, even in negligible way. Often these interactions result in the adopting a common rhythm. How many times have we seen vee formation of birds wavings their wings with the same frequency?

This adjustment of rhythms due to interactions is intuitively the essence of the phenomenon of synchronization, a phenomenon that will be the main subject of this study. [5]

One of the first that examined the phenomenon of synchronization was Christiaan Huygens. He watched the two pendulum clocks hung on the common beam. As a result of the vibrations transmitted by this beam from one clock to the other, after a while they started to move with the same frequency. It is worth mentioning that also Rayleigh and van der Pol whom had a significant contribution to the understanding and developing of this theory. [6]

In this work will be considered a couple of multistable systems. System multistability means that for fixed values of the parameters, it has many

coexisting attractors. The described behavior is a natural characteristic of dynamic systems and can be found in almost every field of natural sciences. [1]
Coupling several of such systems in one common issue, by including in equations the relevant factors, we can observe a very interesting behaviors. Our main priority is to observe the above-mentioned synchronization.

The subject of the studies is van der Pol-Duffing oscillator. The structure is based on the equation of van der Pol oscillator, designed by the aforementioned physicist Balthasar van der Pol. Originally used in the studies of electrical vacuum tubes was later used in many other areas of physical and biological sciences, such as neuroscience or even seismology. [11]
Modification of the equation by adding an external driving force and nonlinear Duffing type stiffness leads to the equation known as the van der Pol-Duffing oscillator equation.

At first basic mathematical definitions and theorems that are necessary for further considerations will be introduced. They will be later expanded in the next chapters. Then, the single van der Pol oscillator dynamics will be examined and later the Duffing modification. In next chapter we will focus on the behavior of the system of 10 coupled oscillators and changes of this behavior under the influence of some actions. Finally, conclusions will be presented, and other modifications that can be considered for this problem.

Chapter 2

Preliminaries

At the beginning the basic concepts and theorems from theory of ordinary differential equations and theory of dynamical systems will be introduced.

2.1 Theory of ordinary differential equations

Let us consider the first-order ordinary differential equation

$$y' = g(t, y), \qquad y(t_0) = y_0, \tag{2.1}$$

where $g : \mathbb{R} \times \mathbb{R}^n \supset U \to \mathbb{R}^n$ is a continuous function, U is an open set and $(t_0, y_0) \in U$.

Definition 2.1.1 *Problem (2.1) is called the initial value problem (the Cauchy problem). [7]*

Definition 2.1.2 *Every function $y^* : I \to \mathbb{R}^n$ of class C^1, where $I \subset \mathbb{R}$ such that for all $t \in I$, $(t, y^*(t)) \in U, (y^*)'(t) = g(t, y^*(t))$ and $y^*(t_0) = y_0$ is called the solution of problem (2.1). [7]*

Definition 2.1.3 *We say that the solution $y^{**} : I_1 \to \mathbb{R}^n$ of (2.1) is an extension of the solution $y^* : I \to \mathbb{R}^n$ of this problem, if $I \subset I_1$ and $y^{**}|I = y^*$. If $I \neq I_1$, then this extension is called the proper one. The solution, for which there is no proper extension is called a global solution. [7]*

Remark 2.1.1 *Let* $y := (y_1, y_2, \ldots, y_n), g := (g_1, g_2, \ldots, g_n), y_0 := (y_0^1, y_0^2, \ldots, y_0^n)$. *Clearly, the problem (2.1) can be represented as the system of n first-order ordinary differential equations, each of the* $\mathbb{R}$ *space. We have*

$$y_i' = g_i(t, y_1, y_2, \ldots, y_n),\ y_i(t_0) = y_0^i, \qquad i = 1, \ldots, n. \tag{2.2}$$

Definition 2.1.4 *We say that the function* g *satisfies on set* $U' \subset U$ *the Lipschitz condition with respect to the variable* y*, if there exist a constant* $L > 0$ *such that for any points* $(t, y^1), (t, y^2) \in U'$ *the condition* $\|g(t, y^1) - g(t, y^2)\| \leq L\|y^1 - y^2\|$ *is satisfied, where* $\|\cdot\|$ *is the norm in space* $\mathbb{R}^n$*. [7]*

Definition 2.1.5 *We will say that function* g *satisfies on set* U *the local Lipschitz condition with respect to the variable* y*, if every point* $(t, y^0) \in U$ *has a neighborhood in which* g *satisfies the Lipschitz condition with respect to the variable* y*. [7]*

Theorem 2.1.1 *Function* g *satisfies on set* $U' \subset U$ *the Lipschitz condition with respect to the variable* y *if and only if each function* $g_1, \ldots, g_n$ *satisfies on* U' *this condition with respect to* y*. [7]*

Proof. Let $(t, y^1), (t, y^2) \in U'$.
" $\Leftarrow$ ": Assume that there exist constant $L > 0$ such that $\|g(t, y^1) - g(t, y^2)\| \leq L\|y^1 - y^2\|$. Let for setted $i \in \{1, \ldots, n\}$, $L_i := L$. Using the Euclidean norm in $\mathbb{R}^n$ and assumption we obtain
$\|g_i(t, y^1) - g_i(t, y^2)\| \leq \sqrt{\sum_{i=1}^n \|g_i(t, y^1) - g_i(t, y^2)\|^2} = \|g(t, y^1) - g(t, y^2)\| \leq L\|y^1 - y^2\| = L_i\|y^1 - y^2\|$.
From the arbitrary on i functions $g_1, \ldots, g_n$ satisfy on U' the Lipschitz condition with respect to the variable y.
" $\Rightarrow$ ": Assume that for every $i \in \{1, \ldots, n\}$ there exist constant $L_i > 0$ such that $0 \leq \|g_i(t, y^1) - g_i(t, y^2)\| \leq L_i\|y^1 - y^2\|$. Let $L := \sqrt{\sum_{i=1}^n L_i^2}$. Using the Euclidean norm and assumption we have
$\|g(t, y^1) - g(t, y^2)\| = \sqrt{\sum_{i=1}^n \|g_i(t, y^1) - g_i(t, y^2)\|^2} \leq \sqrt{\sum_{i=1}^n L_i^2\|y^1 - y^2\|^2} = \|y^1 - y^2\|\sqrt{\sum_{i=1}^n L_i^2} = L\|y^1 - y^2\|$.
So g satisfies on U' the Lipschitz condition with respect to the variable y, which had to be demonstrated. □

Remark 2.1.2 *Similarly, one can show that above theorem remains valid also for the local Lipschitz condition.*

Theorem 2.1.2 *Let $I_1, \ldots, I_n$ be a 1-dimensional intervals and let $\tilde{U} = \mathbb{R} \times I_1 \times \ldots \times I_n$. Assume that function $\tilde{g} : \tilde{U} \to \mathbb{R}^n$, where $(t, y_1, \ldots, y_n) \in \tilde{U}$ is differentiable for the variables $y_1, \ldots, y_n$ on set $\tilde{U}$ and its partial derivatives $\frac{\partial \tilde{g}}{\partial y_i}$ are bounded functions that satisfy*

$$\|\frac{\partial \tilde{g}}{\partial y_i}(t, y_1, \ldots, y_n)\| \leq M_i \quad \text{for every } (t, y_1, \ldots, y_n) \in \tilde{U}, \text{ where } i = 1, \ldots, n.$$

Then, for any two points $a = (t, a_1, \ldots, a_n), b = (t, b_1, \ldots, b_n) \in \tilde{U}$ we obtain $\|f(b) - f(a)\| \leq \sum_{i=1}^{n} M_i \, |b_i - a_i|$. ([7], with proof)

Theorem 2.1.3 *If every function g_i, $i = 1, \ldots, n$ has on set U continuous partial derivatives $\frac{\partial g_i}{\partial y_j}$, $j = 1, \ldots, n$ with respect to the point (t, y), then the function g satisfies on set U the local Lipschitz condition with respect to the variable y. [7]*

Proof. We will consider in $\mathbb{R}^n$ the taxicab norm, that is, for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\|x\| = \sum_{i=1}^{n} |x_i|$.
Let $i \in \{1, \ldots, n\}, (t, y^0) \in U$. The set U is open, so there exist $\delta > 0$ such that the closed ball of radius δ centered at a point (t, y^0) is contained in U, i.e. $\bar{K}((t, y^0), \delta) \subset U$. Because we consider the taxicab metric, so this ball is a Cartesian product of $n + 1$ closed intervals from $\mathbb{R}$. From the compactness of the set $\bar{K}((t, y^0), \delta)$ and assumption about continuous partial derivatives $\frac{\partial g_i}{\partial y_i}, j = 1, \ldots, n$ we have
$\sup \left\{ \|\frac{\partial g_i}{\partial y_i}(\tilde{t}, \tilde{y})\| : (\tilde{t}, \tilde{y}) \in \bar{K}((t, y^0), \delta) \right\} = M_j < \infty, j = 1, \ldots, n.$
Let $L := \max \{M_j : j \in \{1, \ldots, n\}\}$ and let us take any $(t, y^{0,1}) = (t, y_1^{0,1}, \ldots, y_n^{0,1})$, $(t, y^{0,2}) = (t, y_1^{0,2}, \ldots, y_n^{0,2}) \in \bar{K}((t, y^0), \delta)$. From theorem (2.1.2) we have
$\|g_i(t, y^{0,1}) - g_i(t, y^{0,2})\| \leq \sum_{j=1}^{n} M_j \left| y_j^{0,1} - y_j^{0,2} \right| \leq L \sum_{j=1}^{n} \left| y_j^{0,1} - y_j^{0,2} \right| = L \|y^{0,1} - y^{0,2}\|$.
Therefore, g_i satisfies on $\bar{K}((t, y^0), \delta)$ the local Lipschitz condition with respect to the y, and more, on neighborhood $K((t, y^0), \delta) \ni (t, y^0)$. From the arbitrary on $i \in \{1, \ldots, n\}$ and theorem (2.1.1), function g also satisfies on

$K((t, y^0), \delta)$ this condition with respect to the y. From the arbitrary on point $(t, y^0) \in U$, g meets on U local Lipschitz condition with respect to the y (definition (2.1.5)), which had to be demonstrated. □

The above theorem is easy to check sufficient condition for the function g to be locally Lipschitz. This property of function is very important in fundamental theorem of existence and uniqueness of the solution of initial value problem (2.1), which is shown below.

Theorem 2.1.4 *If function g is continuous and satisfies on set U local Lipschitz condition with respect to the variable y, then for every $(t_0, y_0) \in U$ exists uniqueness global solution of (2.1). ([7], with proof)*

Theorem 2.1.5 *If the assumptions of the theorem (2.1.4) are fulfilled, then the solution of (2.1) is continuous function of the initial condition y_0. ([7], with proof)*

Since in the main part of the paper are considered second-order equations, so it is necessary to present a method of bringing the high-order equations into a system of first-order equations.

Let us consider the n-order ordinary differential equation:

$$z^{(n)} = h(t, z, z', \ldots, z^{(n-1)}),\ z(t_0) = z_0^0, z'(t_0) = z_0^1, \ldots, z^{(n-1)}(t_0) = z_0^{n-1}, \tag{2.3}$$

where $h : \mathbb{R} \times \mathbb{R}^n \supset V \to \mathbb{R}$ is a continuous function with continuous partial derivatives with respect to the variables $z, z', \ldots, z^{(n-1)}$, V is an open set and $(t_0, z_0^0, \ldots, z_0^{n-1}) \in V$.

Remark 2.1.3 *It should be noted that in the considered problem (2.3) the solution z is a function that operates in space $\mathbb{R}$, contrary to solution from (2.1), which operates in $\mathbb{R}^n$. The more general case can be considered with z also operating in $\mathbb{R}^n$, but such simplified formulation of the problem in this paper is sufficient.*

Theorem 2.1.6 *Problem (2.3) is equivalent to the problem:*

$$\begin{cases} z_1'(t) = z_2(t),\ z_1(t_0) = z_0^0 \\ z_2'(t) = z_3(t),\ z_2(t_0) = z_0^1 \\ \vdots \\ z_{n-1}'(t) = z_n(t),\ z_{n-1}(t_0) = z_0^{n-2} \\ z_n'(t) = h(t, z_1, z_2, \ldots, z_n),\ z_n(t_0) = z_0^{n-1} \end{cases} \tag{2.4}$$

Proof. Let us put the following substitution. Let for every t from domain of function z: $z_1(t) := z(t), z_2(t) := z'(t), \ldots, z_n(t) := z^{(n-1)}(t)$. Then we have

$z_1'(t) = z'(t) = z_2(t)$

$z_2'(t) = z''(t) = z_3(t)$

$\vdots$

$z_{n-1}'(t) = z^{(n-1)}(t) = z_n(t)$

$z_n'(t) = z^n(t) = h(t, z(t), z'(t), \ldots, z^{(n-1)}(t)) = h(t, z_1(t), z_2(t), \ldots, z_n(t))$

Also:

$z_1(t_0) = z(t_0) = z_0^0$

$\vdots$

$z_n(t_0) = z^{(n-1)}(t_0) = z_0^{n-1}$

Hence we obtain the thesis. □

The theory derived for the (2.1) translates into problem (2.4). Substituting $y_1 := z_1, \ldots, y_n := z_n,\ g_1 := z_2, \ldots, g_{n-1} := z_n, g_n := h(t, z_1, \ldots, z_n)$ and $y_0^1 := z_0^0, \ldots, y_0^n := z_0^{n-1}$ it is easy to convert (2.4) to (2.1). Indeed,

$y' = (y_1', \ldots, y_n') = (z_1', \ldots, z_n') = (z_2, \ldots, z_{n-1}, h(t, z_1, z_2, \ldots, z_n)) = (g_1, \ldots, g_n) = g(t, y)$.

$y(t_0) = (y_1(t_0), \ldots, y_n(t_0)) = (z_1(t_0), \ldots, z_n(t_0)) = (z_0^0, \ldots, z_0^{n-1}) = (y_0^1, \ldots, y_0^n) = y_0$.

What is more, assumptions made for function h are sufficient for g to satisfy assumptions of theorem (2.1.4). Indeed, for every $i \in \{1, \ldots, n-1\}, j \in \{1, \ldots, n\}$, $\frac{\partial g_i}{\partial z_j} = 0$ when $j \neq i + 1$ and $\frac{\partial g_i}{\partial z_j} = 1$ when $j = i + 1$, so these partial derivatives are continuous as constant functions. Continuity $\frac{\partial g_n}{\partial z_j}$ is the result of continuity of function h partial derivatives. According to theorem (2.1.3), the assumptions of (2.1.4) ale fullfiled.

Definition 2.1.6 *Let $\varphi : [t_0, \infty) \to \mathbb{R}^n$ be a solution of problem (2.1). We say that it is stable in the sense of Lyapunov, if for every $\epsilon > 0$ exists $\delta > 0$ such that every solution ψ of this system (for another initial value y_0) that satisfies $|\psi(t_0) - \varphi(t_0)| < \delta$ is determined over the interval $[t_0, \infty)$ and $|\psi(t) - \varphi(t)| < \epsilon$ for all $t \geq t_0$. [8]*

Definition 2.1.7 *We say that the solution $\varphi : [t_0, \infty) \to \mathbb{R}^n$ of problem (2.1) is asymptotically stable, if it is stable and there exist $\delta_0 > 0$ such that for every solution $\psi : [t_0, \infty) \to \mathbb{R}^n$ that satisfies $|\psi(t_0) - \varphi(t_0)| < \delta_0$ we obtain $\lim_{t \to +\infty} |\psi(t) - \varphi(t)| = 0$. [8]*

Definition 2.1.8 *We say that solution φ is unstable, if it is not stable.*

Remark 2.1.4 *Using the substitution $\tilde{y} := y - \varphi$, where φ is the solution of problem (2.1), the stability of φ is equivalent to stability of zero-solution $\tilde{y} \equiv \theta$ of equation*
$\tilde{y}' = (y - \varphi)' = y' - \varphi' = g(t, y) - g(t, \varphi) = g(t, \tilde{y} + \varphi) - g(t, \varphi)$.
Function $\tilde{y} \equiv \theta$ naturally meet the above equation.

2.2 Theory of dynamical systems

In the next part we will be considering the autonomous differential equations, which are independent of the time variable t.
Let us consider therefore problem

$$y' = g(y), \qquad y(t_0) = y_0, \tag{2.5}$$

where $g : \mathbb{R}^n \supset U \to \mathbb{R}^n$ is function that satisfies assumptions of the theorem of existence and uniqueness (2.1.4), U is open set and $y_0 \in U$. We will further assume that the solution of (2.5) is defined on the whole half-line $[t_0, \infty)$. For a fixed $y_0 \in U$, we denote this solution as φ_{y0}.

Definition 2.2.1 *Dynamical system in* $\mathbb{R}^n$ *is called any family* $\{S_t : t \geq 0\}$ *of maps from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$ *such that*
(i) $S_0 = id_{\mathbb{R}^n}$
(ii) $S_t \circ S_\tau = S_{t+\tau}$, *for every* $t, \tau \geq 0$. *[10]*

Remark 2.2.1 *In addition, we assume that the map* $[0, +\infty) \times \mathbb{R}^n \ni (t, y) \mapsto S_t(y)$ *is continuous.*

Solutions of differential equations define some dynamical systems. Declares that the following theorem.

Theorem 2.2.1 *Let for* $y_0 \in U$, $S_{t-t_0}(y_0)$ *be the value of the solution* φ_{y0} *in while* $t \geq t_0$. *Family* $\{S_{t-t_0} : t - t_0 \geq 0\}$ *is a dynamical system in set* $U \subset \mathbb{R}^n$.

Proof. The continuity of the map $(\tilde{t}, y) \mapsto S_{\tilde{t}(y)}$ follows from the continuity of solution (2.5) with respect to variable $\tilde{t}$ and from theorem (2.1.5) about continuous relationship of solution from the initial condition.
We check the definition (2.2.1) conditions.
(i) Let $y_0 \in U$. Then $S_0(y_0) = S_{t_0-t_0}(y_0) = y(t_0) = y_0$.
(ii) Let $y_0 \in U, t, \tau \geq 0$. According to uniqueness of the solution of problem (2.5) we obtain
$(S_t \circ S_\tau)(y_0) = S_t(S_\tau(y_0)) = S_{(t+t_0)-t_0}(S_{(\tau+t_0)-t_0}(y_0)) = S_{(t+t_0)-t_0}(y(\tau+t_0)) = y(\tau+t_0+t+t_0) = y((\tau+t+t_0)+t_0) = S_{(\tau+t+t_0)-t_0}(y(t_0)) = S_{t+\tau}(y_0)$. Hence we obtain the thesis. □

Basing on the above, the whole theory of dynamical systems can be considered in relation to the autonomous differential equations. Below are introduced the basic concepts and theorems of the theory.

Definition 2.2.2 *Image of solution* φ_{y0}, *i.e. set* $\{\varphi_{y_0}(t) : t \geq t_0\}$ *is called the trajectory of the system (2.5), starting from initial condition* y_0. *[10]*

Uniqueness of the solution indicates that the trajectories starting from different initial conditions can not intersect with each other.
There are several major types of trajectories.

Definition 2.2.3 1° *Trajectory is called the fixed point when* φ_{y_0} *is constant function.*
2° *Trajectory is called the periodic trajectory when* φ_{y_0} *is periodic function.*
3° *Trajectory is called the quasi-periodic trajectory when* φ_{y_0} *is quasi-periodic function. [10]*

Definition 2.2.4 *We say that the function* $u : \mathbb{R} \to \mathbb{R}$ *is quasi-periodic when it is of the form* $u(t) = F(\alpha_1 t, \ldots, \alpha_n t)$ *for some* $n \in \mathbb{N}$*, where* $F : \mathbb{R}^n \to \mathbb{R}$ *is function* 2π*-periodic with respect to each variable and numbers* $\alpha_1, \ldots, \alpha_n$ *are independent over the field of rational numbers, i.e. they satisfy the condition* $\sum_{i=1}^{n} a_i \alpha_i = 0,\ a_i \in \mathbb{Q},\ i = 1, \ldots, n\ \Rightarrow a_i = 0,\ i = 1, \ldots, n.$

The fixed points are the trajectories, which are the simplest to find. We only have to solve the equation $g = \theta$ with respect to the variable y. Finding other types of trajectories is much more difficult.

Definition 2.2.5 *Figure of the trajectories with selected direction of movement on them is called a phase portrait of the system. [10]*

Let us return to the concept of the stability of solutions for differential equations. Because in a fairly simple way we can find the trajectories that are fixed points, so it would be important to have tools to investigate their stability. It turns out that such tools exist - these are the Lyapunov stability theorems. They relate to the zero-solutions, but by virtue of remark (2.1.4) test of stability of other fixed points can be reduced to test of zero-solution, using the appropriate substitution. Let us assume, therefore, that the system (2.5) has a zero-solution $y \equiv \theta$, i.e. $g(\theta) = \theta$.

Theorem 2.2.2 *If there exist function (called Lyapunov function)* $V : U \to [0, \infty)$ *of class* C^1 *vanishing only in* θ *and such that function* $\dot{V}(y) := \langle g(y), V'(y) \rangle \le 0$ *for every* $y \in U$*, where* $\langle \cdot, \cdot \rangle$ *is scalar product in* $\mathbb{R}^n$*, then the zero-solution of problem (2.5) is stable. ([8], with proof)*

Theorem 2.2.3 *If there exist function $V : U \to [0, \infty)$ of class C^1 vanishing only in θ and such that function $\dot{V}(y) := \langle g(y), V'(y) \rangle < 0$ for every $y \in U \setminus \{\theta\}$, then the zero-solution of problem (2.5) is asymptotically stable. ([8], with proof)*

Theorem 2.2.4 *If there exist function (called Lyapunov anti-function) $V : U \to [0, \infty)$ of class C^1 vanishing only in θ and such that function $\dot{V}(y) := \langle g(y), V'(y) \rangle > 0$ for every $y \in U \setminus \{\theta\}$ and there exist sequence $(x_n)_{n \in \mathbb{N}}$ that is convergent to θ and such that for every $n \in \mathbb{N}$, $V(x_n) > 0$, then the zero-solution of problem (2.5) is unstable. ([8], with proof)*

At the end of this chapter two theorems that are used to study the existence of periodic trajectories of differential equations on the plane are presented. First, presented by Poincare and Bendixon, let us show the existence of such trajectory for any equation on the plane, but its assumptions are usually difficult to verify. The second - Lienard theorem, however, narrows the class of equations, but it is very easy to handle.

Definition 2.2.6 *Let $y_0 \in U$ be fixed. The ω-limit set for point y_0 is called set $\omega(y_0) := \{\lim_{n \to +\infty} \varphi_{y_0}(t_n) : t_n \to +\infty \ for \ n \to +\infty\}$. [10]*

Theorem 2.2.5 *Let in the problem (2.5) $n := 2$ and let $y_0 \in U$ be fixed. If set $\{\varphi_{y_0}(t) : t \geq t_0\}$ is a bounded set on the plane and its closure does not contain fixed points, then $\omega(y_0)$ is periodic trajectory. [10]*

Theorem 2.2.6 *Let us consider the second-order differential equation*

$$z'' + L_1{}'(z)z' + L_2(z) = 0, \tag{2.6}$$

where L_2 is continuous function and L_1 is function of class C^1 (equations of this form are called the Lienard equations). Assume that (2.6) meets the assumptions of theorem (2.1.4). If:
(i) L_1, L_2 are odd functions,
(ii) $L_2(z) > 0$ for $z > 0$,

(iii) $L_1(z) \to +\infty$ *for* $|z| \to +\infty$ *and there exist constant* $\beta_1 > 0$ *such that for* $z > \beta_1$ *function* L_1 *is positive and increases monotonically,*
(iv) there exist constant $\beta_0 > 0$ *such that for* $z \in (0, \beta_0), L_1(z) < 0$*,*
then equation (2.6) has periodic solution. In addition, if $\beta_0 = \beta_1$*, then that solution is unique and asymptotically stable. [9]*

Proof of the Lienard theorem, using Poincare-Bendixon theorem (2.2.5), can be found in [9].

Chapter 3

Van der Pol - Duffing oscillator

In this chapter, we examine the dynamics of the two types of oscillators. First, we consider the van der Pol oscillator and second, Duffing modification. Examining the second one will provide a starting point for further studies.

3.1 Van der Pol oscillator

Let us consider the initial value problem

$$x'' - \alpha(1 - x^2)x' + x = 0, \qquad x(0) = y_1, x'(0) = y_2, \tag{3.1}$$

where $x : [0, \infty) \to \mathbb{R},\ \alpha \geq 0,\ y_1, y_2 \in \mathbb{R}$.

Definition 3.1.1 *System (3.1) is called the problem of van der Pol unforced oscillator. x is a function of the position of oscillator in time and α is parameter. [1]*

Before analyzing the dynamics, let us transform the system (3.1) to a system of two first-order equations. We will use both of them interchangeably.

Theorem 3.1.1 *Problem (3.1) is equivalent to the problem*

$$\begin{cases} x_1' = x_2,\ x_1(0) = y_1 \\ x_2' = \alpha(1 - x_1^2)x_2 - x_1,\ x_2(0) = y_2, \end{cases} \tag{3.2}$$

where $x_1, x_2 : [0, \infty) \to \mathbb{R},\ \alpha \geq 0,\ y_1, y_2 \in \mathbb{R}$

Proof. We will use the theorem (2.1.6). Rewriting the equation to the form that is in the theorem we have $x'' = \alpha(1 - x^2)x' - x$. In this case $n = 2$ and function h is the right side of this equation. From theorem (2.1.6) we obtain

$$\begin{cases} x_1'(t) = x_2(t),\ x_1(0) = y_1 \\ x_2'(t) = \alpha(1 - x_1(t)^2)x_2(t) - x_1(t),\ x_2(0) = y_2 \end{cases} \tag{3.3}$$

Hence we obtain the thesis. □

Remark 3.1.1 *For further needs let us denote* $\dot{f}(x_1, x_2) := (x_2, \alpha(1-x_1^2)x_2 - x_1)$.

Theorem 3.1.2 *System (3.2) has exactly one global solution for any fixed* $y_1, y_2 \in \mathbb{R}$.

Proof. Let $U = [0, \infty] \times \mathbb{R}^2$ and let $f_1, f_2 : U \to \mathbb{R}, f : U \to \mathbb{R}^2$ be denoted as $f_1(t, x_1, x_2) := x_2,\ f_2(t, x_1, x_2) := \alpha(1 - x_1^2)x_2 - x_1,\ f(t, x_1, x_2) := (f_1(t, x_1, x_2), f_2(t, x_1, x_2))$. Functions f_1, f_2, f are of course continuous on U with respect to every variable as elementary functions. Furthermore
$\frac{\partial f_1}{\partial x_1} = 0, \quad \frac{\partial f_1}{\partial x_2} = 1,$
$\frac{\partial f_2}{\partial x_1} = -2\alpha x_1 x_2 - 1, \quad \frac{\partial f_2}{\partial x_2} = \alpha(1 - x_1^2).$
These partial derivatives treated as functions of the variable (t, x_1, x_2) on set U are also continuous as elementary functions. Therefore from theorem (2.1.3) f_1, f_2 satisfy on set U local Lipschitz condition with respect to variable (x_1, x_2) and from theorem (2.1.1) and remark (2.1.2) the same concition on U satisfies also function f. Hence, by theorem (2.1.4) system (3.2) has exactly one global solution for any fixed $y_1, y_2 \in \mathbb{R}$, which had to be demonstrated. □

Theorem 3.1.3 *If* $\alpha = 0$*, then the problem (3.1) has explicit form of solution, given by formula* $\varphi(t) = y_1 \cos t + y_2 \sin t$.

Proof. We present the solving method that uses the characteristic equation. Description of the method, which allows to solve linear second-order differential equations can be found in [9].

Equation in the present case is of the form $x^2+x=0$. Hence, the characteristic equation is $\lambda^2+1=0$. Its roots are $\lambda_1=i, \lambda_2=-i$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. So the solution of output problem is the function $\varphi(t)=e^0(C_1 y_1 \cos t + C_2 y_2 \sin t) = C_1 y_1 \cos t + C_2 y_2 \sin t$, where $C_1, C_2 \in \mathbb{R}$. Using the initial conditions (3.1) we have

$y_0 = \varphi(0) = C_1 + 0 = C_1$

$y_1 = \varphi'(0) = -\sin(0)C_1 + \cos(0)C_2 = C_2$.

Therefore $\varphi(t) = y_1 \cos t + y_2 \sin t$, which had to be demonstrated. □

Beneath is a chart of an exemplary solution of (3.1), where $\alpha = 0$.

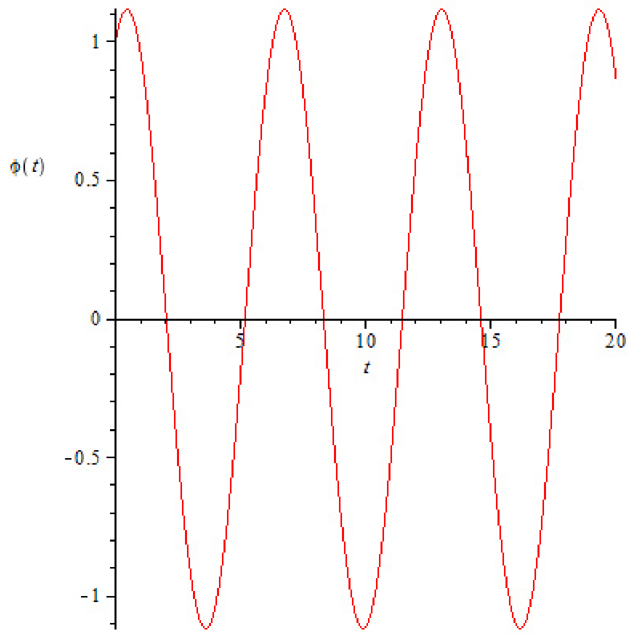

Figure 3.1.1 *Solution of (3.1) for* $y_1 = 1, y_2 = 0.5$.

In the following section, we assume that $\alpha > 0$.

Theorem 3.1.4 *System (3.2) has exactly one fixed point* $(x_1^*, x_2^*) = (0,0)$.

Proof. Equating the right sides of equations from (3.2) to zero we obtain

$$\begin{cases} x_2 = 0 \\ \alpha(1-x_1^2)x_2 - x_1 = 0 \end{cases}$$

Therefore
$x_2 = 0$ and $(1 - x_1^2)0 - x_1 = 0 \Rightarrow x_1 = 0$.
Thus, the only fixed point is $(x_1^*, x_2^*) = (0, 0)$. Hence we obtain the thesis. □

Theorem 3.1.5 *Fixed point $(x_1^*, x_2^*) = (0, 0)$ of system (3.2) is unstable for $x_1 \in (-1, 1), x_2 \in \mathbb{R}$.*

Proof. According to theorem (2.2.4), let $W := (-1, 1) \times \mathbb{R}$. Then $(x_1^*, x_2^*) \in U$. Let us consider the function $V : W \to \mathbb{R}$ given by formula $V(x_1, x_2) := \frac{1}{2\alpha}x_1^2 + \frac{1}{2\alpha}x_2^2$. We will show that V is the Lyapunov anti-function of system (3.2). Indeed,
1) V is of class C^1 and $\nabla V(x_1, x_2) = [\frac{1}{\alpha}x_1, \frac{1}{\alpha}x_2]$.
2) $V(x_1, x_2) = 0 \Leftrightarrow \frac{1}{2\alpha}x_1^2 + \frac{1}{2\alpha}x_2^2 = 0 \Leftrightarrow (x_1, x_2) = (0, 0)$.
3) $\dot{V}(x_1, x_2) = (\dot{f}(x_1, x_2)|\nabla V(x_1, x_2)) = x_2\frac{1}{\alpha}x_1 + (\alpha x_2 - \alpha x_1^2 x_2 - x_1)\frac{1}{\alpha}x_2 = \frac{1}{\alpha}x_1 x_2 + x_2^2 - x_1^2 x_2^2 - \frac{1}{\alpha}x_1 x_2 = x_2^2(1 - x_1^2) > 0$, because $(x_1, x_2) \in U$.
4) Let us consider sequence $x_n := (\frac{1}{n}, \frac{1}{n})$ for $n \in \mathbb{N}$. Then $\lim_{n\to\infty} x_n = (0, 0)$ and $V(x_n) = \frac{1}{2\alpha}(\frac{1}{n})^2 + \frac{1}{2\alpha}(\frac{1}{n})^2 = \frac{1}{\alpha}(\frac{1}{n})^2 > 0$ for every $n \in \mathbb{N}$.
Hence we obtain that system (3.2) has Lyapunov anti-function and by theorem (2.2.4) the zero-solution $(x_1^*, x_2^*) = (0, 0)$ is unstable fixed point for $(x_1, x_2) \in U$, which had to be demonstrated. □

Theorem 3.1.6 *System (3.2) has asymptotically stable unique periodic solution.*

Proof. We will use theorem (2.2.6). Note, that equation from (3.2) is a special case of the equation from this theorem. Indeed, here
${L_1}'(x) := -\alpha(1 - x^2), L_2(x) := x$, function L_2 is of course continuous and function $L_1(x) = -\alpha x + \frac{1}{3}\alpha x^3$ is of class C^1. Note that,
(i') $L_1(-x) = \alpha x - \frac{1}{3}\alpha x^3 = -(-\alpha x + \frac{1}{3}\alpha x^3) = -L_1(x)$
(i'') $L_2(-x) = -x = -L_2(x)$, so functions L_1, L_2 are odd.
(ii) Let $x > 0$. Then $L_2(x) = x > 0$.
(iii) $\lim_{|x|\to\infty} L_1(x) = \lim_{|x|\to\infty} x^3(-\frac{\alpha}{x^2} + \frac{1}{3}\alpha) = +\infty$. Furthermore, let $\beta_1 := \sqrt{3}$ and $x > \beta_1$. Hence $L_1(x) = -\alpha x(1 - \frac{\sqrt{3}}{3}x)(1 + \frac{\sqrt{3}}{3}x) > 0$ and $L_1'(x) =$

$-\alpha(1-x)(1+x) > 0$, so L_1 is positive and strictly increasing for $x \in (\beta_1, \infty)$.
(iv) Let $\beta_0 := \sqrt{3}$ and $x \in (0, \beta_0)$. Then $L_1(x) = -\alpha x(1-\frac{\sqrt{3}}{3}x)(1+\frac{\sqrt{3}}{3}x) < 0$.
What is more, $\beta_0 = \beta_1$.
Hence, from theorem (2.2.6), system (3.2) has unique periodic solution and it is asymptotically stable, which had to be demonstrated. □

Beneath are the charts of a phase portrait of problem (3.2).

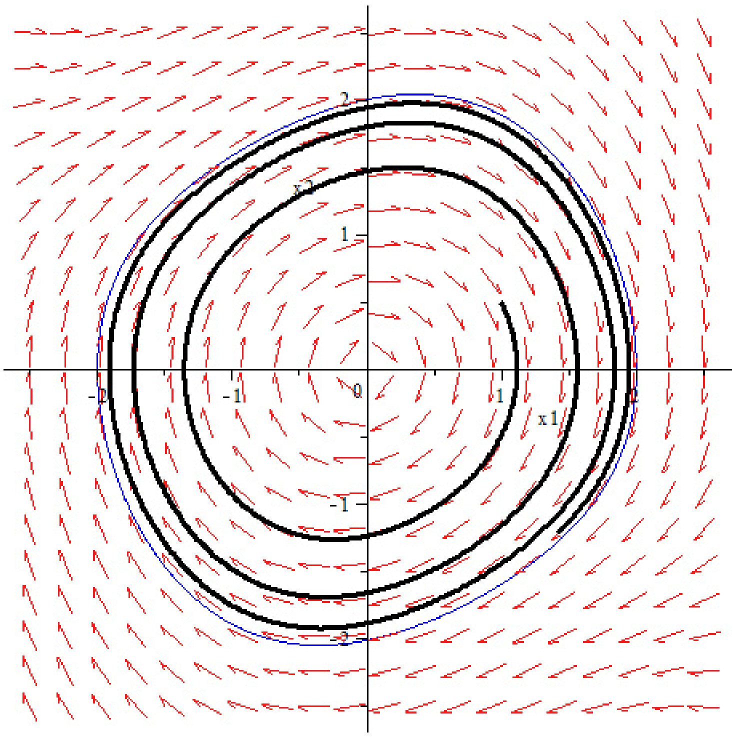

Figure 3.1.2 *Phase portrait of (3.2),* $y_1 = 1, y_2 = 0.5, \alpha = 0.2$.

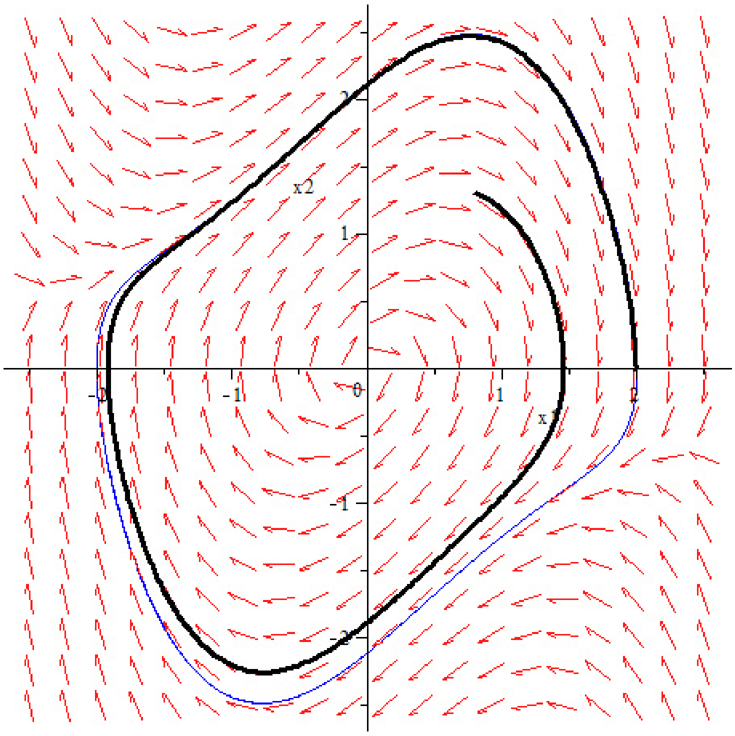

Figure 3.1.3 *Phase portrait of (3.2),* $y_1 = 0.8, y_2 = 1.3, \alpha = 0.8$.

The black curve is the trajectory and the blue one is the attractor, in this case periodic solution. As we can see, the fixed point repels other trajectories and the periodic trajectory attracts them.
The shape of the solution is highly dependent on the parameter value α. For small values the solution is sinusoidal, but for bigger we can observe the relaxation oscillations. It means that it tends to resemble a series of step functions, jumping between positive and negative values twice per cycle. Detailed study of this can be found in [11].

3.2 Van der Pol - Duffing oscillator

In this section, we examine the dynamics of a single van der Pol - Duffing oscillator. The results will be very useful in studies about major issue, that is in next chapter.

Remark 3.2.1 *In this and the following part the results are obtained by the numerical calculations. The method that was used is fourth-order Runge-Kutta method of numerical soluting the ordinary differential equations. Description of the algorithm can be found in [9].*

Let us consider the initial value problem

$$x'' - \alpha(1 - x^2)x' + x^3 = F\sin(\omega t), \qquad x(0) = y_1, x'(0) = y_2, \tag{3.4}$$

where $x : [0, \infty) \to \mathbb{R},\ \alpha, F, \omega > 0,\ y_1, y_2 \in \mathbb{R}$.

Definition 3.2.1 *Problem (3.4) is called the problem of van der Pol - Duffing oscillator. As before, x is a function of the position of oscillator in time and α, F, ω are parameters. [1]*

Theorem 3.2.1 *Problem (3.4) is equivalent to the problem*

$$\begin{cases} x_1' = x_2,\ x_1(0) = y_1 \\ x_2' = \alpha(1 - x_1^2)x_2 - x_1^3 + F\sin(\omega t),\ x_2(0) = y_2, \end{cases} \tag{3.5}$$

where $x_1, x_2 : [0, \infty) \to \mathbb{R},\ \alpha, F, \omega > 0,\ y_1, y_2 \in \mathbb{R}$

Proof. Again we use theorem (2.1.6). Rewriting the equation to the form that is in the theorem we obtain $x'' = \alpha(1 - x^2)x' - x^3 + F\sin(\omega t)$. In this case $n = 2$ and function h is the right side of this equation. From theorem (2.1.6) we have

$$\begin{cases} x_1'(t) = x_2(t),\ x_1(0) = y_1 \\ x_2'(t) = \alpha(1 - x_1(t)^2)x_2(t) - x_1(t)^3 + F\sin(\omega t),\ x_2(0) = y_2 \end{cases} \tag{3.6}$$

Hence we obtain the thesis. □

Theorem 3.2.2 *System (3.5) has exactly one global solution for any fixed $y_1, y_2 \in \mathbb{R}$.*

Proof. Let $U = [0, \infty] \times \mathbb{R}^2$ and let $f_1, f_2 : U \to \mathbb{R}$, $f : U \to \mathbb{R}^2$ be denoted as $f_1(t, x_1, x_2) := x_2$, $f_2(t, x_1, x_2) := \alpha(1 - x_1^2)x_2 - x_1^3 + F\sin(\omega t)$, $f(t, x_1, x_2) := (f_1(t, x_1, x_2), f_2(t, x_1, x_2))$. Functions f_1, f_2, f are of course continuous on U with respect to every variable as elementary functions. Furthermore
$\frac{\partial f_1}{\partial x_1} = 0, \quad \frac{\partial f_1}{\partial x_2} = 1$
$\frac{\partial f_2}{\partial x_1} = -2\alpha x_1 x_2 - 3x_1^2, \quad \frac{\partial f_2}{\partial x_2} = \alpha(1 - x_1^2)$
These partial derivatives treated as functions of the variable (t, x_1, x_2) on set U are also continuous as elementary functions. Therefore from theorem (2.1.3) f_1, f_2 satisfy on set U local Lipschitz condition with respect to variable (x_1, x_2) and from theorem (2.1.1) and remark (2.1.2) the same condition on U satisfies also function f. Hence, by theorem (2.1.4) system (3.2) has exactly one global solution for any fixed $y_1, y_2 \in \mathbb{R}$, which had to be demonstrated.
□

Remark 3.2.2 *It should be noted that the equation from (3.4) is not an autonomous equation, i.e. it depends on time variable t.*
Conversion (3.4) to the system of autonomous equation is related to additional function $x_0(t) := t$ and third equation in system - $x_0' = 1$. According to the theory of dynamical systems, formally we should use this new form of the system. But this operation, however, does not bring any new information, and further complicates the problem, increasing the dimension of the system from 2. to 3. That is why it will not be changed. But keep in mind that the trajectories drawn on the plane (x, x') are projections of the trajectories from $\mathbb{R}^3$.

Remark 3.2.3 *In the following sections it is assumed that $\alpha = 0.2, F = 1, \omega = 0.962$.*

We will introduce some of the more advanced definitions of the theory of dynamical systems, which have not been needed so far. We also present their applications in considered problem.
Modify the system (2.5). Let us consider the dynamical system given by

initial value problem

$$y' = g(y, \gamma),\ y(0) = y_0 \tag{3.7}$$

where $g : \mathbb{R}^n \supset U \times \Gamma \subset \mathbb{R} \to \mathbb{R}^n, y_0 \in U$ and $\gamma \in \Gamma$ is parameter. Assume that g satisfies assumptions of theorem (2.1.4) and for every $y_0 \in U$ it has a solution φ_{y_0} defined on the whole half-line $[0, \infty)$. Denote $\varphi_{y_0} := (\varphi^1_{y_0}, \dots \varphi^n_{y_0})$.

Definition 3.2.2 *Set $A \subset \mathbb{R}^n$ is called positively invariant set of the system (3.7) if for every $y_0 \in A$ and $t \geq 0$, $\varphi_{y_0}(t) \in A$. [10]*

Definition 3.2.3 *Compact set $A \subset \mathbb{R}^n$ is called local attractor of system (3.7) if it satisfies the following conditions*
1° *A is positively invariant;*
2° *There exist its neighbourhood $A^* \subset \mathbb{R}^n$ such that*

$$\lim_{t\to\infty} (\sup \{d(\varphi_{y_0}(t), A) : y_0 \in A^*\}) = 0, \tag{3.8}$$

where $d(w, Z) := \inf \{d(w, z) : z \in Z\}$ is the distance between the point w and the set Z, given by metric $\mathbb{R}^n$;
3° *For any set $A' \subset A$, if A' satisfies the conditions 1° and 2°, then $A' = A$. [10]*

Definition 3.2.4 *If in the above definition, the condition (3.8) is safistied for every set $A^* \subset \mathbb{R}^n$, then set A is called the global attractor. [10]*

Definition 3.2.5 *We say that a dynamical system is multistable, if there exist more than one local attractor. [1]*

In multistable systems very important is the initial condition, from which the trajectory starts. Depending on this value, the trajectory is attracted by a certain attractor. Closely related to this is the concept of basin of attraction of that attractor.

Definition 3.2.6 *Basin of attraction of the attractor A of system (3.7) is called set $B(A) := \{y_0 \in U : \omega(y_0) \subset A\}$, where $\omega(y_0)$ is ω-limit set for point y_0. [3]*

Remark 3.2.4 *By saying that the system (3.7) is on the attractor A we mean, that* $y_0 \in B(A)$.

Examples of the basins of attraction for considered problem are shown beneath. On horizontal axis are values for y_1 and on vertical axis are values for y_2.

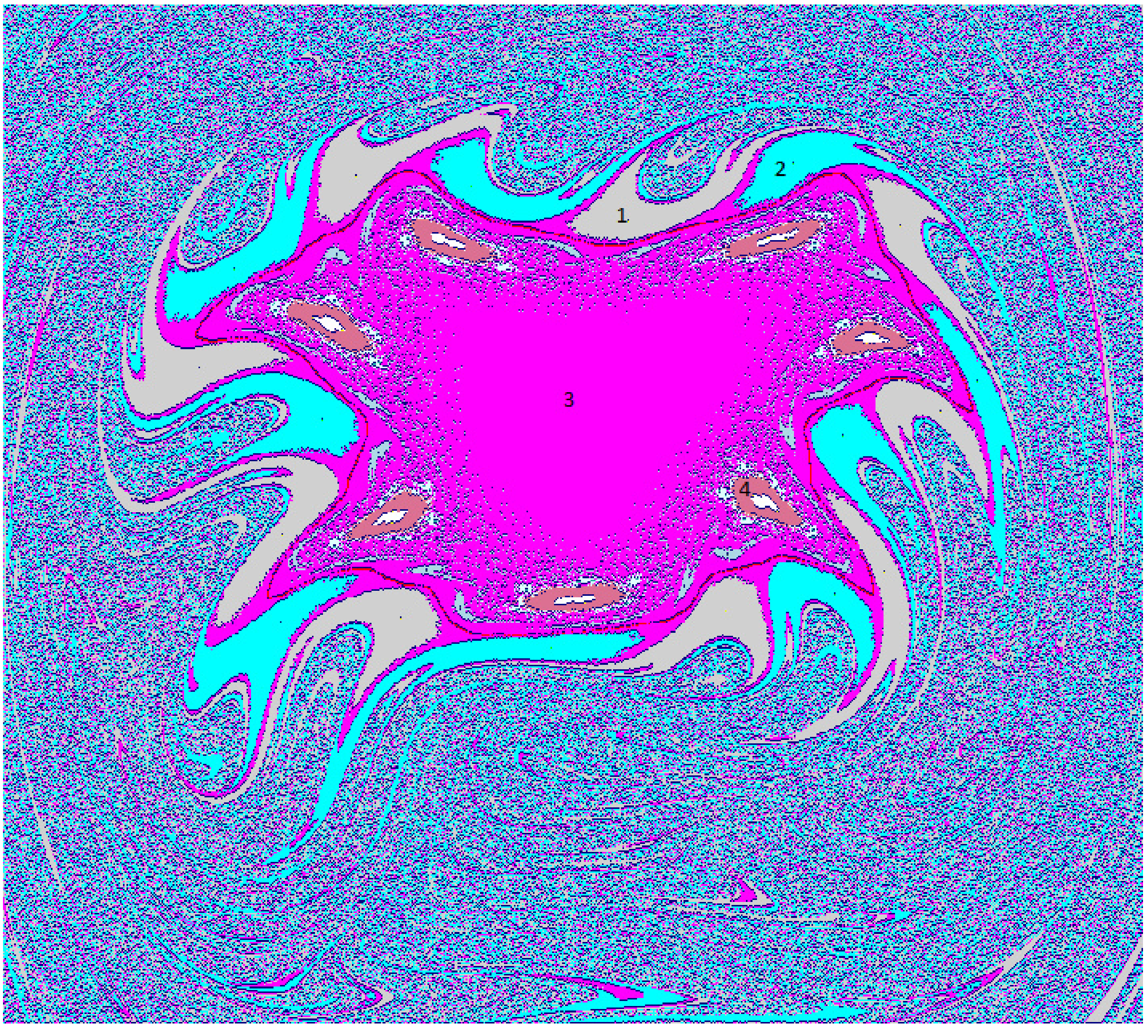

Figure 3.2.1 *Basins of attraction for (3.5),* $y_1 \in [-2, 2], y_2 \in [-2, 3]$.

The different colors represents the basins of attraction for different attractors. We can see that some basins are more extensive than others, which makes the probabilities of hitting a different attractors by selecting random initial values are different. Some basins can be very small, which shows the zoom of the above figure.

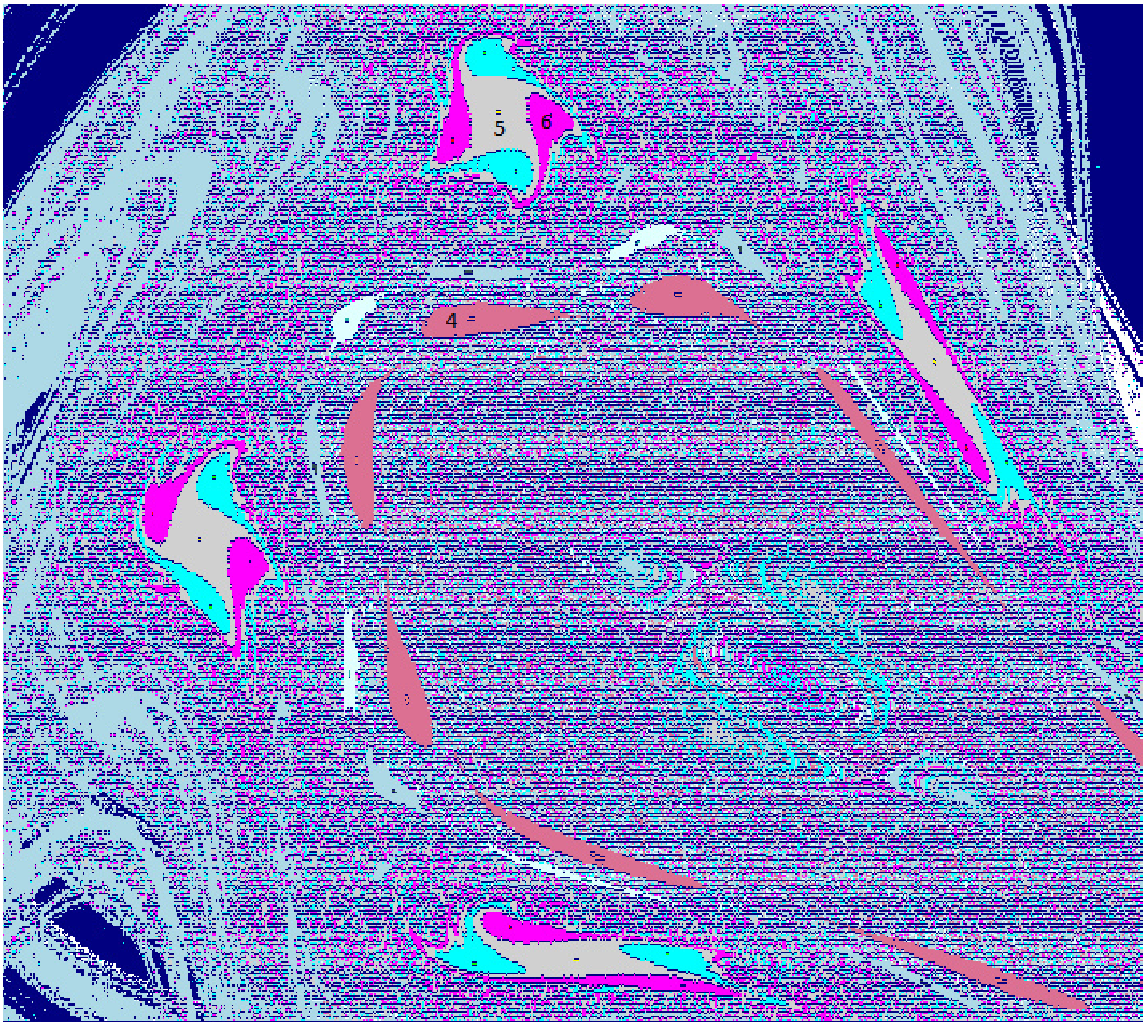

Figure 3.2.2 *Basins of attraction for (3.5),* $x_1 \in [0.48, 0.764]$, $x_2 \in [0.44, 0.765]$.

By bounding the interval of initial values we find an entirely new attractors, as shown in the attached figure.

Theorem 3.2.3 *Let* $\Gamma' \subset \Gamma$ *be a set of parameters such that for* $\gamma \in \Gamma'$ *there exist attractor* A *of system (3.7). Let* $B(A)$ *be a basin of attraction of*

that attractor. Assuming that initial value y_0 and parameter γ are chosen independently, the probability that the system is on the attractor A is equal to
$p(A) := \frac{\lambda(\Gamma')}{\lambda(\Gamma)} \frac{\lambda(B(A))}{\lambda(U)}$,
where λ is measure on $\mathbb{R}^n$. [1]

Definition 3.2.7 *Attractor A is called rare attractor, if $p(A) << 1$. [1]*

Remark 3.2.5 *The choise of sets Γ and U is very important. By manipulating these sets, the rare or non-rare property of attractor can be changed.*

Example 3.2.1 *Referring to the basins of attraction from the last figures. Considering the initial values for (3.2.1) shows, that the basin (4) is so small that the corresponding attractor is surely rare. The more, attractors for basins (5) and (6) on the figure (3.2.2), which on (3.2.1) can not be seen. But if we consider the conditions for (3.2.2), then the attractors for (4), (5) and (6) might not be rare, because the volumes of these basins are bigger.*

More information about the basins of attraction for this problem, but for different parameter values ω can be found in [1].

If we know what are the shapes of the basins of attraction for some attractors, we can choose initial values such that trajectories starting from them are attracted by different attractors.
Let $\{\varphi(t) : t \geq 0\}$ be a trajectory of system (3.5), where $y_1, y_2 \in B(A)$ are fixed and let $t_0 > 0$ be fixed number. For sufficiently large t_0, set $\{\varphi(t) : t \geq t_0\}$ is an approximate picture of the attractor A.

Remark 3.2.6 *Presented graphs are approximate shapes of attractors, projected on the plane (x_1, x_2) (i.e. (x, x')).*

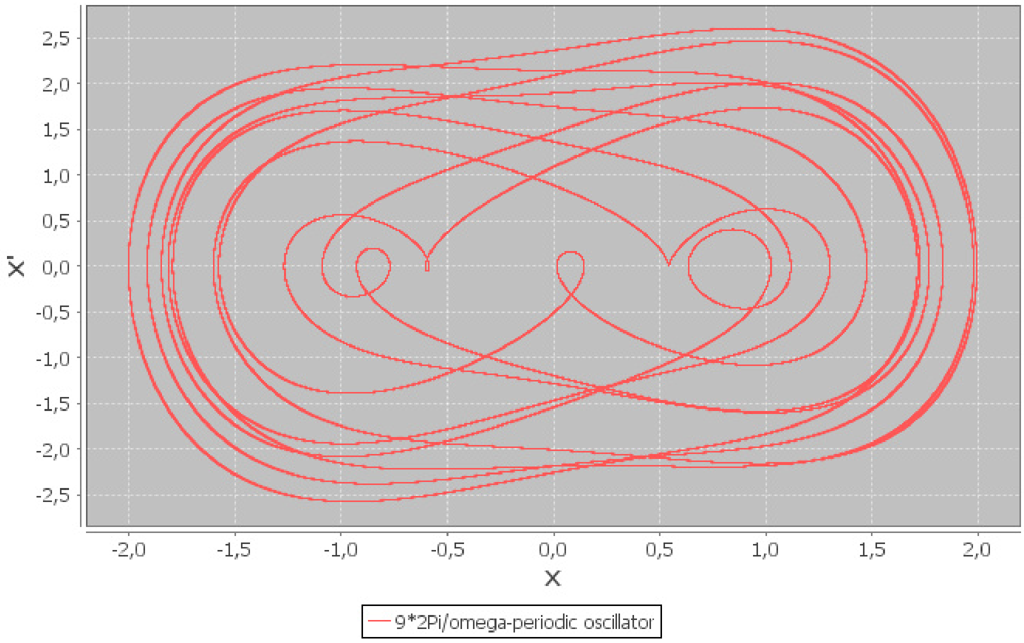

Figure 3.2.3 *Attractor (3.5), conditions* $x_1 = -1.3093855, x_2 = 1.208122$.

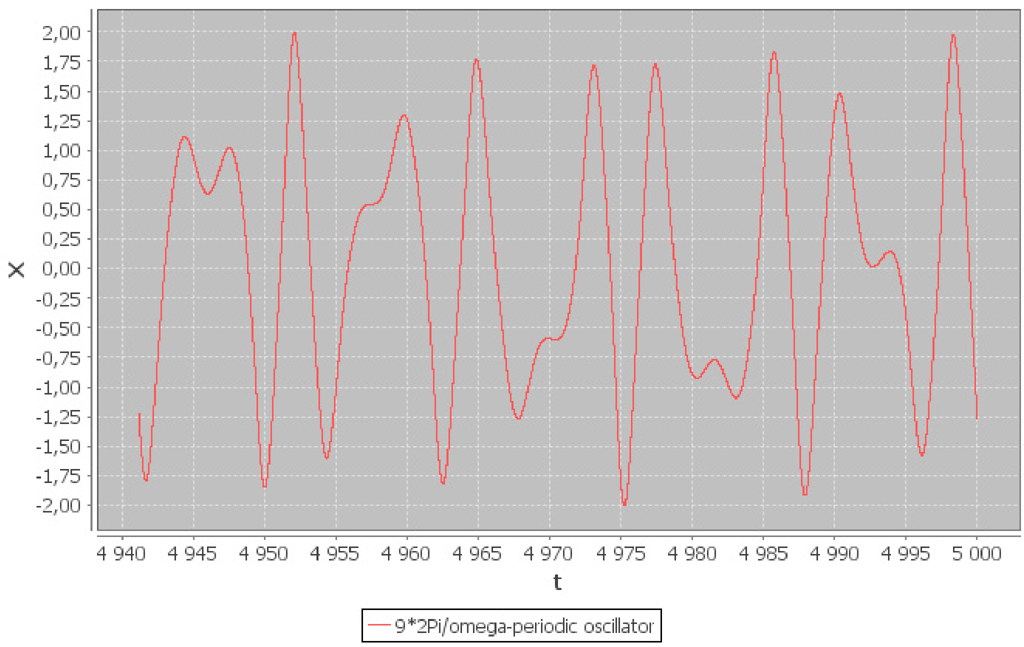

Figure 3.2.4 *Solution (3.5), conditions* $x_1 = -1.3093855, x_2 = 1.208122$.

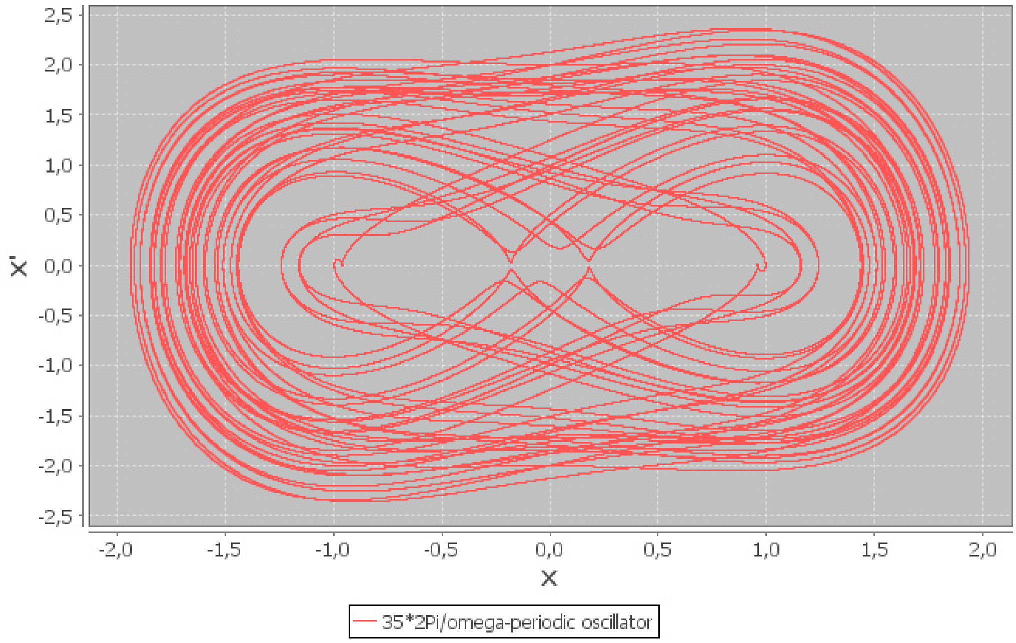

Figure 3.2.5 *Attractor (3.5), conditions* $x_1 = 0.8595725, x_2 = 1.2792925.$

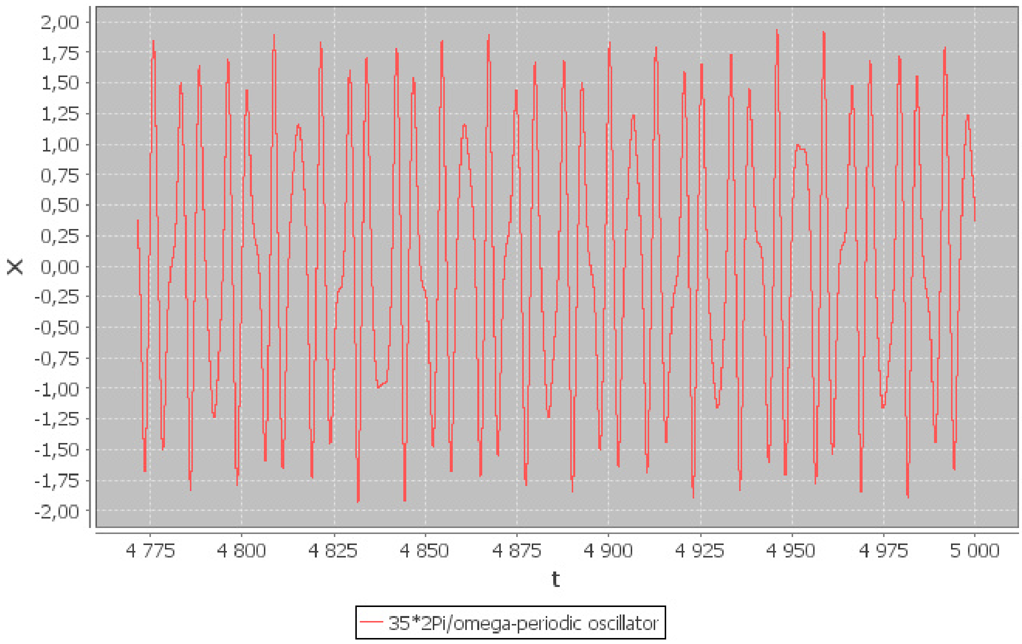

Figure 3.2.6 *Solution (3.5), conditions* $x_1 = 0.8595725, x_2 = 1.2792925.$

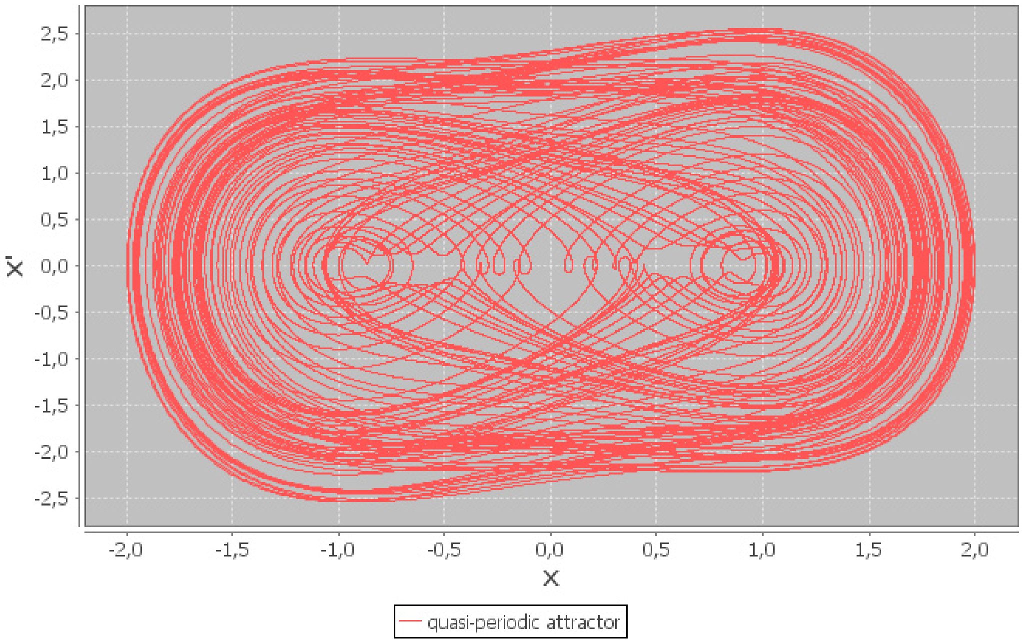

Figure 3.2.7 *Fragment of attractor (3.5),* $x_1 = 0.14585, x_2 = 1.812765.$

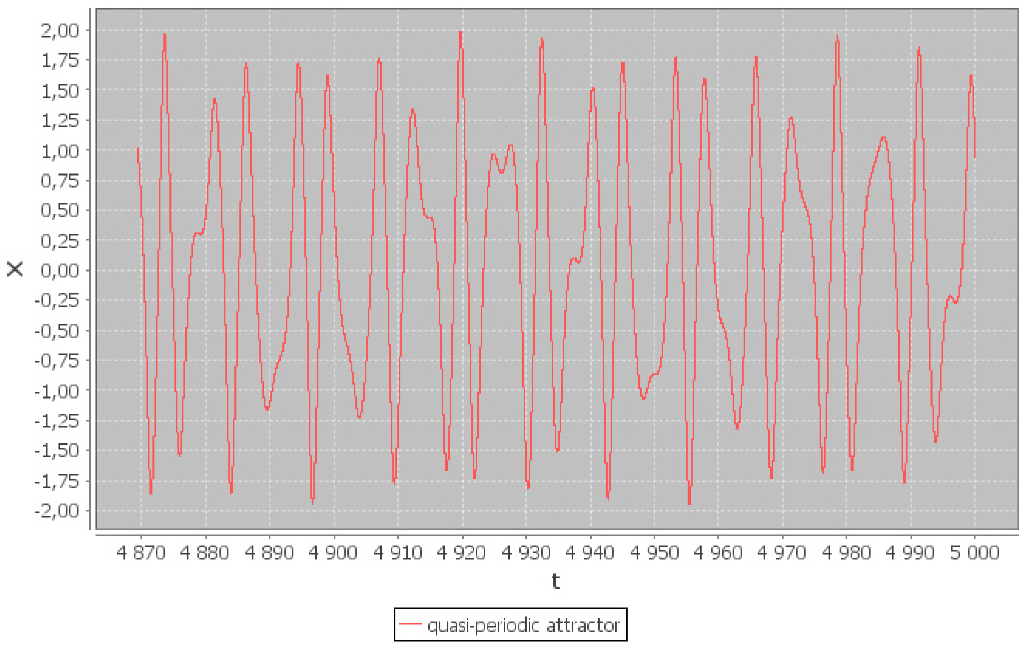

Figure 3.2.8 *Solution (3.5), conditions* $x_1 = 0.14585, x_2 = 1.812765.$

As shown above, some of the curves are closed. This suggests that the attractor is a periodic trajectory. Basing only on the graphs of the trajectories is difficult to determine the exact value of the period. Very helpful in solving this problem is the idea of Poincare map.

Definition 3.2.8 *Let* $n = 2, T \in \mathbb{R}_+, t' > 0$*. For system (3.7), the Poincare map with base period* T *is called set*
$P := \left\{(\varphi^1_{y_0}(kT), \varphi^2_{y_0}(kT)) \in \mathbb{R}^2 : k \in \mathbb{N},\ kT \geq t'\right\}.$
t' *is the time after which we begin to put points on the map. [6]*

Remark 3.2.7 *Poincare maps allow to observe periodic solutions easily and by this to determine the nature of the attractors.* m *points on the map (separated by some neighborhoods) suggests a* mT*-periodic solution. Closed curve suggests a quasiperiodic solution.*

Beneath are the charts of exemplary Poincare maps for system (3.5). The base period is $T = \frac{2\pi}{\omega} \approx 6.531$.

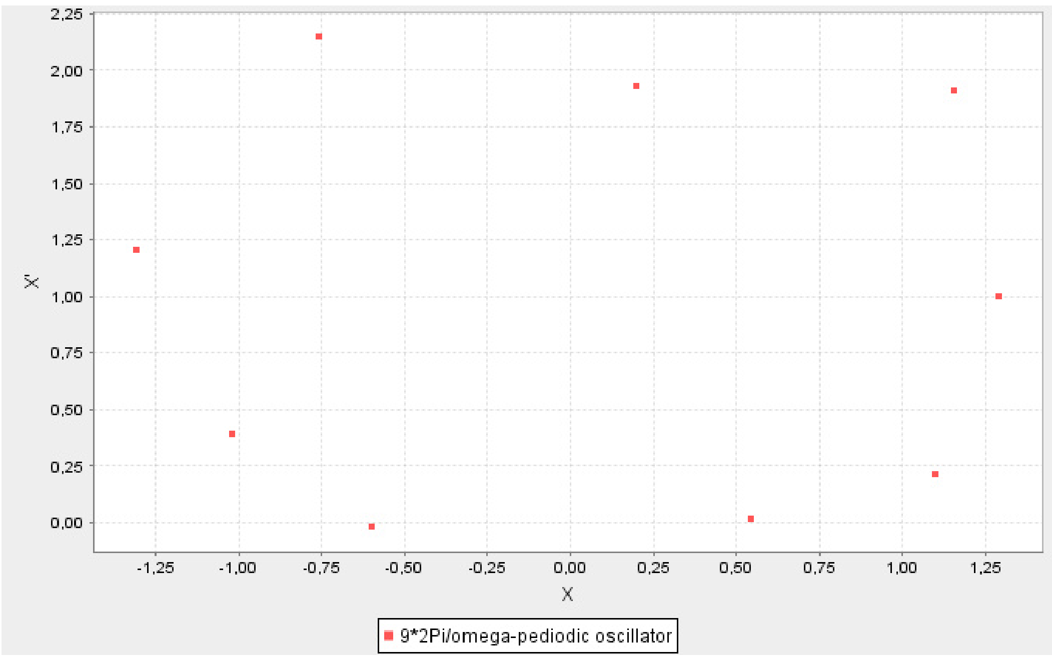

Figure 3.2.9 *Poincare map of (3.5),* $x_1 = -1.3093855, x_2 = 1.208122.$

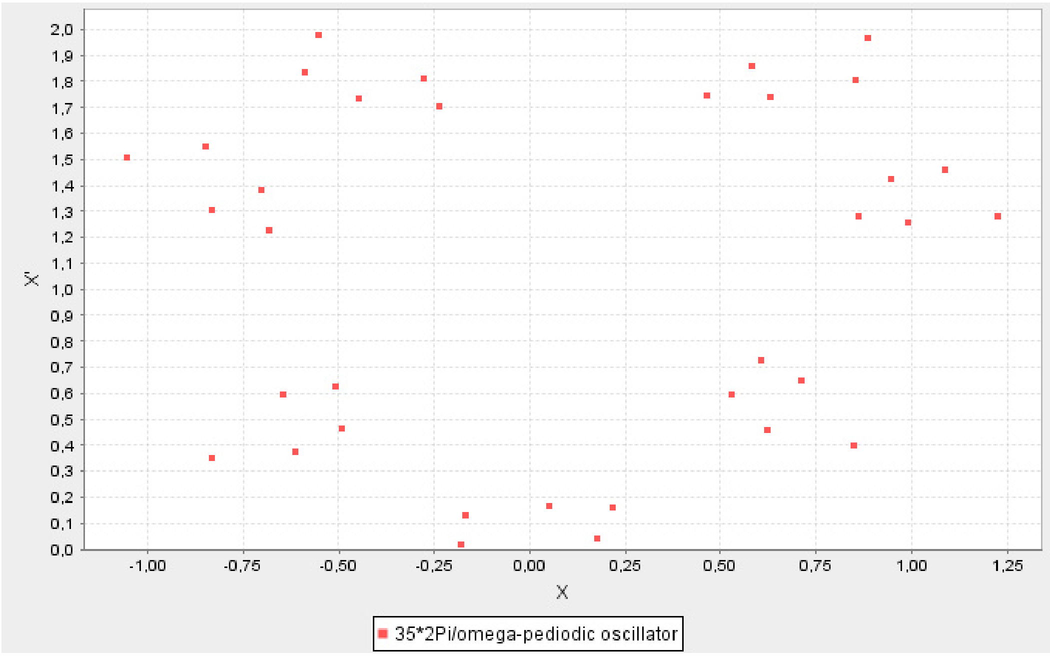

Figure 3.2.10 *Poincare map of (3.5),* $x_1 = 0.8595725, x_2 = 1.2792925.$

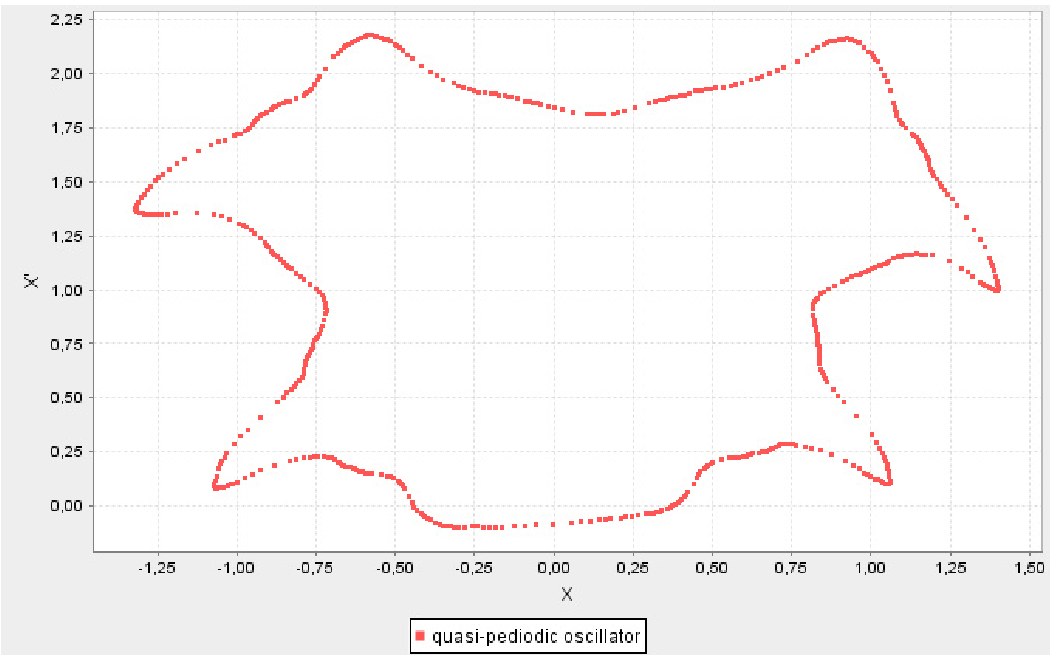

Figure 3.2.11 *Poincare map of (3.5),* $x_1 = 0.14585, x_2 = 1.812765.$

Map (3.2.9) suggests $9T$-periodic trajectory, map (3.2.10) $35T$-periodic and last (3.2.11) quasi-periodic.

Chapter 4

Coupled van der Pol - Duffing oscillators

We already know the dynamics of the single van der Pol - Duffing oscillator. This chapter contains the main issues of this work, the study of the behavior of coupled oscillators.

4.1 Coupled van der Pol - Duffing oscillators

Let there be 10 single van der Pol - Duffing systems

$$\begin{cases} x'_{i,1} = x_{i,2},\ x_{i,1}(0) = y_{i,1} \\ x'_{i,2} = 0.2(1 - x_{i,1}^2)x_{i,2} - x_{i,1}^3 + \sin(0.962t),\ x_{i,2}(0) = y_{i,2}, \end{cases} \tag{4.1}$$

where $i = 1, \ldots, 10,\ x_{i,1}, x_{i,2} : [0, \infty) \to \mathbb{R},\ y_{i,1}, y_{i,2} \in \mathbb{R}$.

Definition 4.1.1 *The system of coupled van der Pol - Duffing oscillators*

(4.1) is called the system

$$\begin{cases} x'_{1,1} = x_{1,2},\ x_{1,1}(0) = y_{1,1} \\ x'_{1,2} = 0.2(1 - x_{1,1}^2)x_{1,2} - x_{1,1}^3 + \sin(0.962t) + \varepsilon(x_{1,1} - x_{10,1}),\ x_{1,2}(0) = y_{1,2} \\ x'_{i,1} = x_{i,2},\ x_{i,1}(0) = y_{i,1} \\ x'_{i,2} = 0.2(1 - x_{i,1}^2)x_{i,2} - x_{i,1}^3 + \sin(0.962t) + \varepsilon(x_{i,1} - x_{i-1,1}),\ x_{i,2}(0) = y_{i,2}, \end{cases} \tag{4.2}$$

where $i = 2, \ldots, 10,\ x_{1,1}, x_{1,2}, x_{i,1}, x_{i,2} : [0, \infty) \to \mathbb{R},\ y_{1,1}, y_{1,2}, y_{i,1}, y_{i,2} \in \mathbb{R}$, $\varepsilon \geq 0$ is the coupling parameter and $t \in [0, \tilde{t}]$ for some $\tilde{t} > 0$ (we consider the problem at some finite time horizon). For $j \in \{1, \ldots, 10\}$, $x_{j,1}$ is a function of the position of j oscillator and $x_{j,2}$ is a function of its speed. In addition, we say that the pairs of oscillators 1 and 10 and $(j+1)$ and j for $j = 1, \ldots, 9$ are coupled.

The method of the coupling of systems comes from [3], where this issue has been discussed for the general case.

Remark 4.1.1 *We assumed the value $\tilde{t} = 5000$.*

Remark 4.1.2 *If $\varepsilon = 0$, equations of the oscillators are independent and the problem (4.2) obviously comes to (4.1).*

From (4.2) (for $\varepsilon > 0$) we can see that the dynamics of each oscillator affects and depends on the dynamics of others. This allows us to observe very rich dynamics of the entire system.
The basic step is to determine the initial values. Let:
$y_{1,1} := -1.3093855, y_{1,2} := 1.208122, y_{2,1} := 0.774456, y_{2,2} := 2.2131965,$
$y_{3,1} := -0.5153313, y_{3,2} := 0.2199307, y_{4,1} := 0.8595725, y_{4,2} := 1.2792925,$
$y_{5,1} := 0.760535, y_{5,2} := 0.54321, y_{6,1} := -0.34133, y_{6,2} := 1.71832,$
$y_{7,1} := -0.8415, y_{7,2} := 1.508125, y_{8,1} := -0.528235, y_{8,2} := 1.98004,$
$y_{9,1} := 0.600255, y_{9,2} := 0.7488, y_{10,1} := 0.14585, y_{10,2} := 1.812765.$
Locations of the oscillators on the attractors are as follows:
1. on $9\frac{2\pi}{0.962}$-periodic

2. on $9\frac{2\pi}{0.962}$-periodic (different from attractor for 1. oscillator)
3. on $25\frac{2\pi}{0.962}$-periodic
4. on $35\frac{2\pi}{0.962}$-periodic
5. on $49\frac{2\pi}{0.962}$-periodic
6. on $49\frac{2\pi}{0.962}$-periodic (different from attractor for 5. oscillator)
7. on $63\frac{2\pi}{0.962}$-periodic
8. on $70\frac{2\pi}{0.962}$-periodic
9. on $70\frac{2\pi}{0.962}$-periodic (different from attractor for 8. oscillator)
10. on quasi-periodic.

The main problem is to explore, depending on what value of the parameter ε oscillators synchronize. Let us introduce a couple of important definitions that have not been used so far.
There are many types of synchronization. In this paper, we focus on the concept of the lag synchronization. The following definitions are directly relevant to the considered system (4.2), but can be easily generalized to the case of any dynamical system.

Definition 4.1.2 *We say that the oscillators $j_1, j_2 \in \{1, \ldots, 10\}$, $j_1 \neq j_2$ are in anti-phase lag synchronization, if there exists $\tau \in \mathbb{R}$ (called lag), $\delta \leq 0.01x^A_{j_2,1}$ and $t_s \in [0, \tilde{t}]$ such that for every $t \geq t_s$ occurs*
$|x_{j_2,1}(t) - (-x_{j_1,1}(t + \tau))| \leq \delta$, where $x^A_{j_2,1} := \max\{|x_{j_2,1}(t)| : t \geq t_s\}$. [2]

Is studies we assumed the value $t_s = \frac{\tilde{t}}{2} = 2500$.

Remark 4.1.3 *The above definition assumes that the demanded accuracy can not be in excess of 1% of the maximum value of the position of the oscillator j_2, but this is not the rule. Depending on your expectations, satisfying may be weaker or stronger assumption for δ.*

Remark 4.1.4 *We briefly say that the oscillators are synchronized if they are in anti-phase lag synchronization.*

τ does not have to be defined uniquely. By changing its value, we can get smaller or larger values of δ, that still satisfies the definition (4.1.2). If the priority is the smalest value for τ, it is enough to find the smallest τ for which δ satisfies (4.1.2). If, however, more important is the accuracy, we should consider a couple values of τ and from received values of δ satisfying (4.1.2) select the minimum. In this thesis, we choosed the second criterion. Next is introduced a helpful function to manage with this problem. The idea of it was taken from [2].

Assume that oscillators j_1, j_2 are synchronized. Let us consider function $\Delta : \mathbb{R} \to [0, \infty)$ given by
$\Delta(\tau) := \max \{|x_{j_2,1}(t) - (-x_{j_1,1}(t+\tau))| : t \geq t_s\}$
Finding the local minima of this function it can be tested for which values of τ, δ will be the smallest. Without loss of generality we consider here the values $\tau \geq 0$.

Definition 4.1.3 *We say that the system (4.2) is synchronized, if any two coupled oscillators are synchronized.*

Remark 4.1.5 *If synchronization occurs for some ε^*, then it occurs also for every $\varepsilon \geq \varepsilon^*$.*

At the beginning the bifurcation diagrams will be presented. Concept of the bifurcation means a sudden, qualitative change in the dynamics of system that occurs when a value of the parameter is changed. [10]
There are many different types of bifurcations that changes the dynamics. Change of the parameter can lead to a period doubling of solutions, changes of its stability or even the existence of new attractors. Here, the most important is the moment of synchronization.
We present a few examples of bifurcation diagrams for the problem (4.2), where ε is the bifurcation parameter. We are trying to see how changes the period of solution for individual oscillators.
Determine $\varepsilon, j \in \{1, \ldots, 10\}$ and $m \in \mathbb{N}$. Denote

$B_{\varepsilon,j} := \{(\varepsilon, x_{j,1}((k+m)\frac{2\pi}{0.962})) \in \mathbb{R}^2 : k \in \mathbb{N} \cup \{0\} \wedge (k+m)\frac{2\pi}{0.962} < \tilde{t}\}$. The bifurcation diagram for j oscillator is the set $Bif_j := \bigcup\{B_{\varepsilon,j} : \varepsilon \geq 0\}$. Points are placed from time value $m\frac{2\pi}{0.962}$. Due to the numerical limitations should be kept in mind that the presented set Bif_j is finite. We consider only a finite number of values for ε, i.e. $\varepsilon \in \{\varepsilon_0, \ldots, \varepsilon_n\}$ for some determined $n \in \mathbb{N}$ and $\varepsilon_l - \varepsilon_{l-1} = const, l = 1, \ldots, n$.

Remark 4.1.6 *Important is to emphasize that with the increase of ε, next considered problem starts from the end points of the trajectory of the previous one. Formally, let for denoted $\varepsilon_l, l \in \{1, \ldots, n\}$ system (4.2) with parameter value $\varepsilon = \varepsilon_l$ starts from initial values $y_{j,1}, y_{j,2}, j = 1, \ldots, 10$. Denote $y_{j,1}^{\varepsilon_l} := x_{j,1}(\tilde{t}), y_{j,2}^{\varepsilon_l} := x_{j,2}(\tilde{t})$. Then for new system (4.2) with parameter value $\varepsilon = \varepsilon_{l+1}$ we consider new initial values $y_{j,1} := y_{j,1}^{\varepsilon_l}, y_{j,2} := y_{j,1}^{\varepsilon_l}, j = 1, \ldots, 10$. This remark applies not only to bifurcation diagrams, but also to other dynamics presentations (for example Poincare maps), which will be presented later in the chapter.*

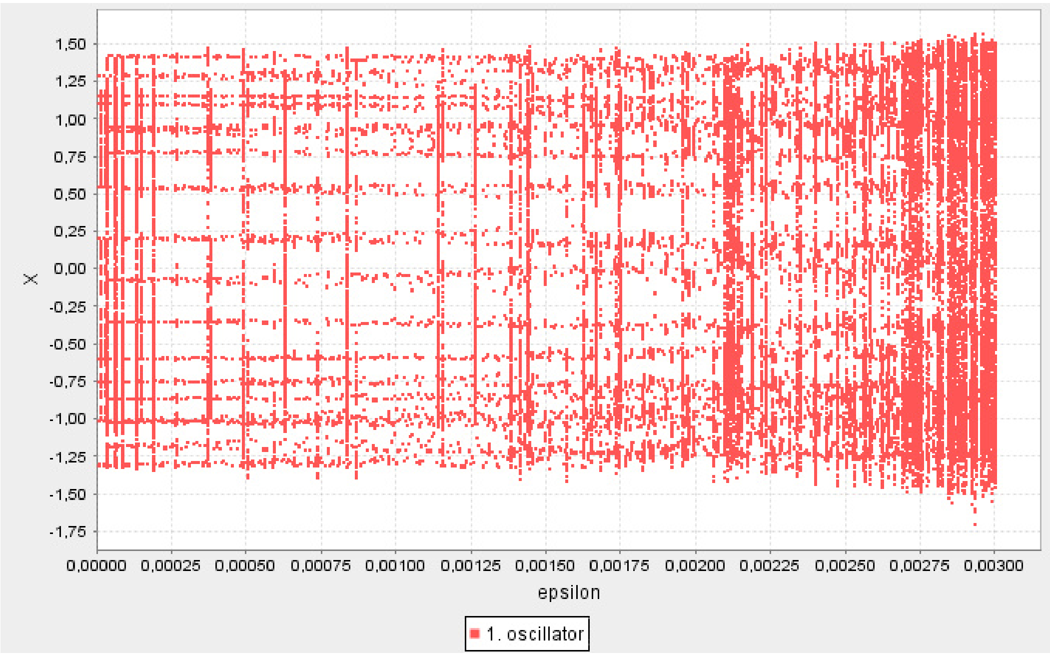

Figure 4.1.1 *Bifurcation diagram of 1. oscillator, $\varepsilon \in [0, 0.003]$.*

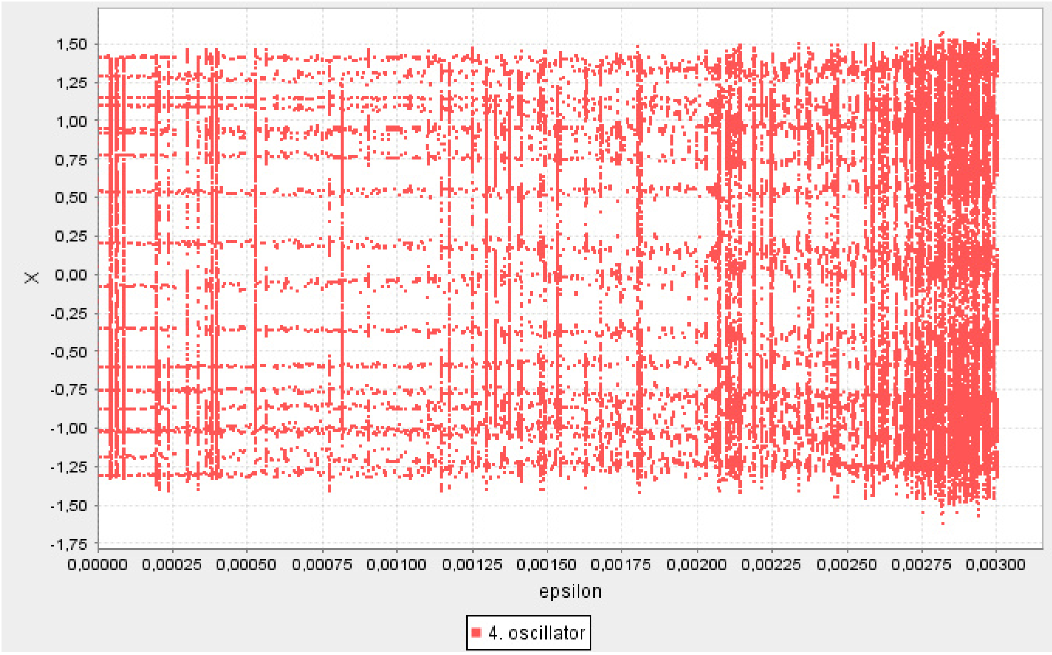

Figure 4.1.2 *Bifurcation diagram of 4. oscillator,* $\varepsilon \in [0, 0.003]$.

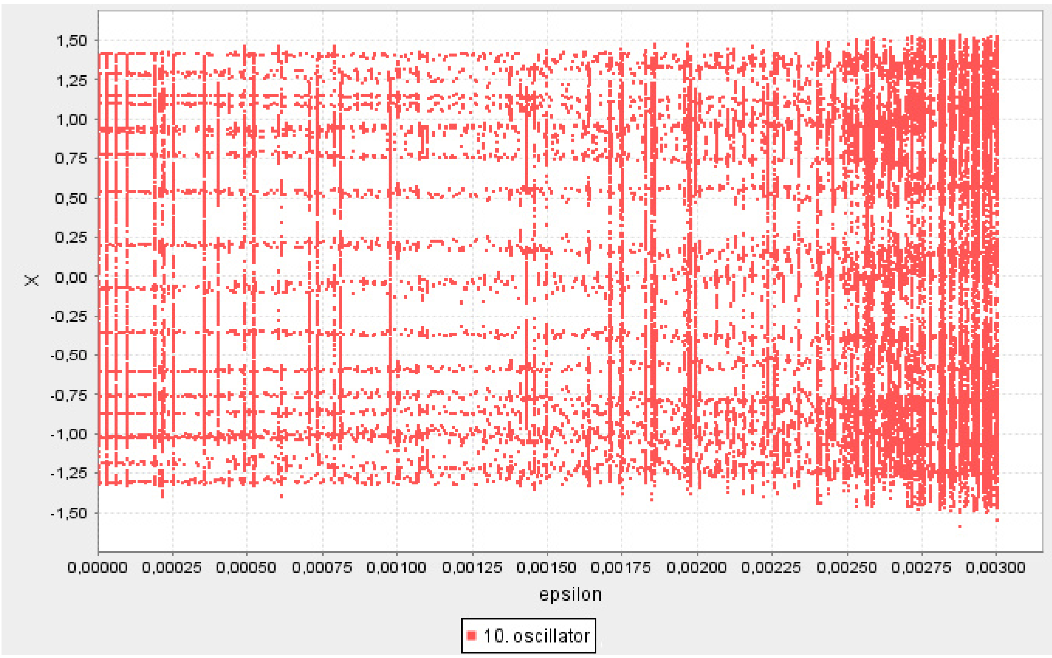

Figure 4.1.3 *Bifurcation diagram of 10. oscillator,* $\varepsilon \in [0, 0.003]$.

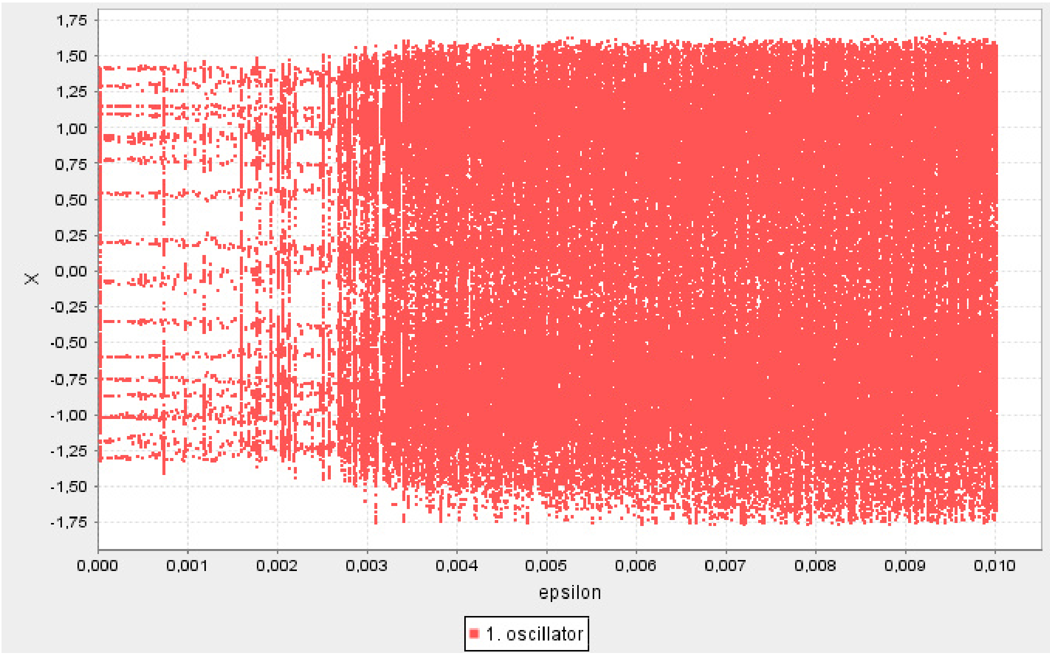

Figure 4.1.4 *Bifurcation diagram of 1. oscillator,* $\varepsilon \in [0, 0.01]$.

As we can see, very small change in the value of the parameter ε greatly disturbs the dynamics of the system. What is more, quite fast ($\varepsilon \approx 0.003$) dynamics of the oscillators begins to suggest chaos or quasiperiodicity (eventually very large period). This can be seen as filling points on the graph above the value of ε parameter, as shown on the figure (4.1.4). This shape of bifurcation diagrams maintaines also for higher values of ε, until the oscillators begin to oscillate periodically, which will be discussed further.

It is worth to mention at this point about the bifurcation diagrams for a single van der Pol - Duffing oscillator, where as the parameter is taken α, F or ω. Very interesting is the case for parameter ω, where depending on its values, appears some completely new attractors, including chaotic. Detailed study of this can be found in [1].

In solving considered problem more helpful are Poincare maps. For a fixed value of ε, their construction is as in the definition (3.2.8) from the previous chapter (the map is created for each oscillator individually).

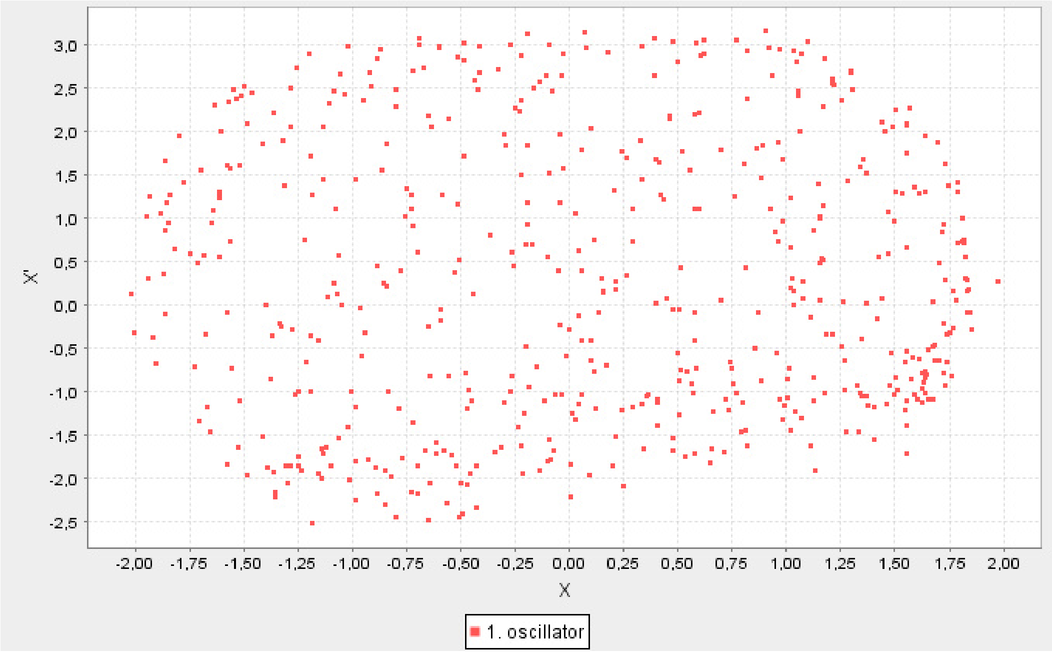

Figure 4.1.5 *Poincare map of 1. oscillator,* $\varepsilon = 0.1$.

For $\varepsilon = 0.1$ map suggests chaos, there is no synchronization.

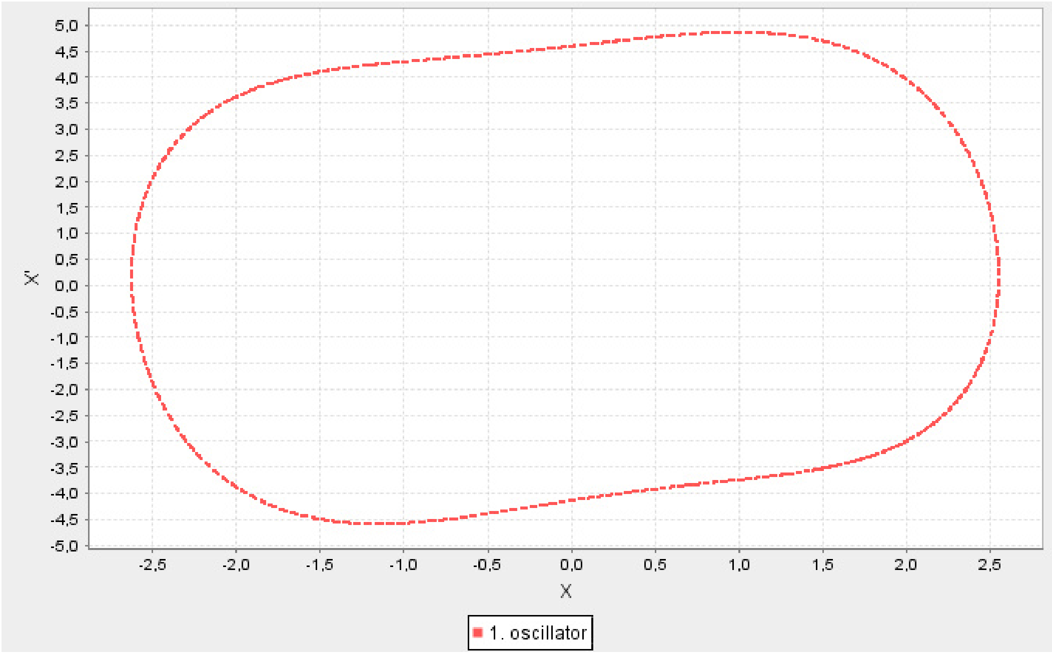

Figure 4.1.6 *Poincare map of 1. oscillator,* $\varepsilon = 0.4$.

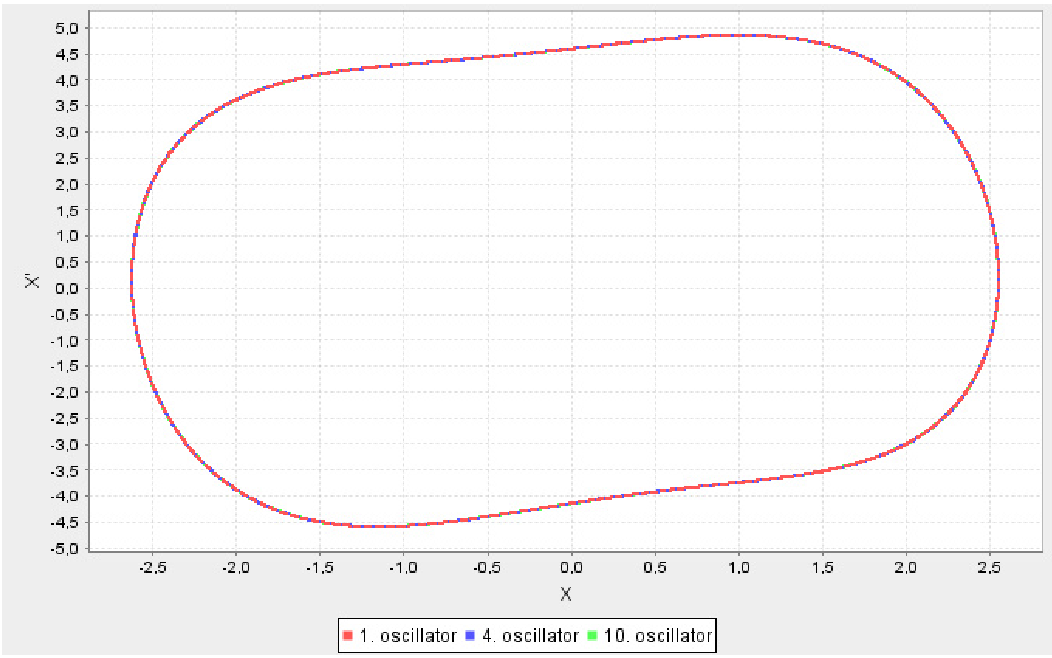

Figure 4.1.7 *Poincare maps for 1., 4. and 10. oscillators,* $\varepsilon = 0.4$.

For $\varepsilon = 0.4$ the maps overlap, so oscillators are on the same quasiperiodic attractor. This occurs not only for oscillators 1., 4. and 10., but also for others.
Beneath the approximate shape of the common attractor (this is, of course, projection on the plane).

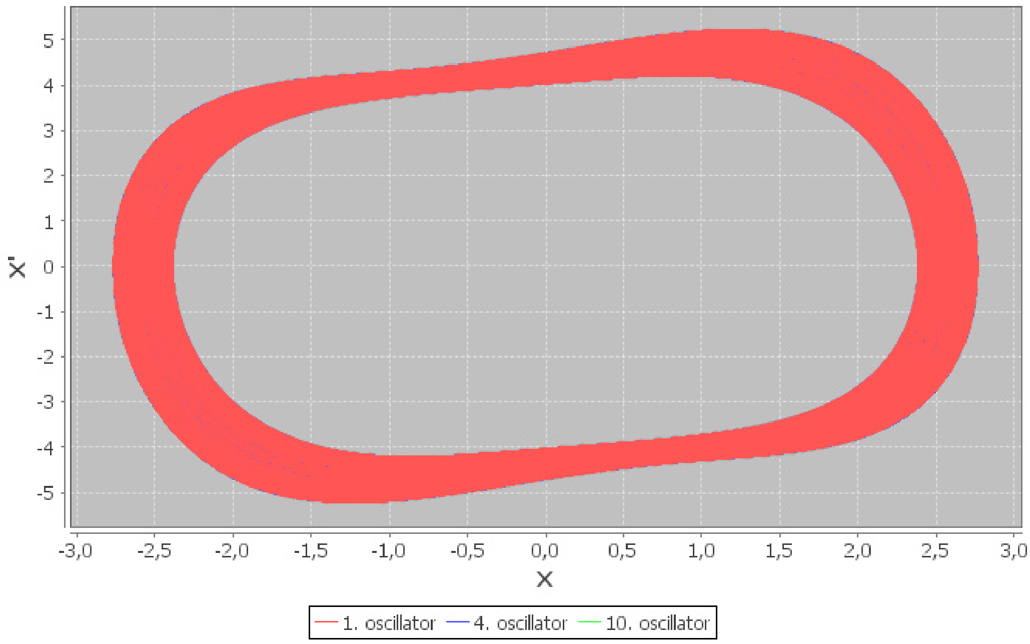

Figure 4.1.8 *Common attractor of 1., 4. and 10. oscillator,* $\varepsilon = 0.4$.

Next are presented the graph of Δ function, synchronization graphs (i.e. the graphs of position of one oscillator to the other) and the position in the time charts. This will illustrate the phenomenon of anti-phase lag synchronization.

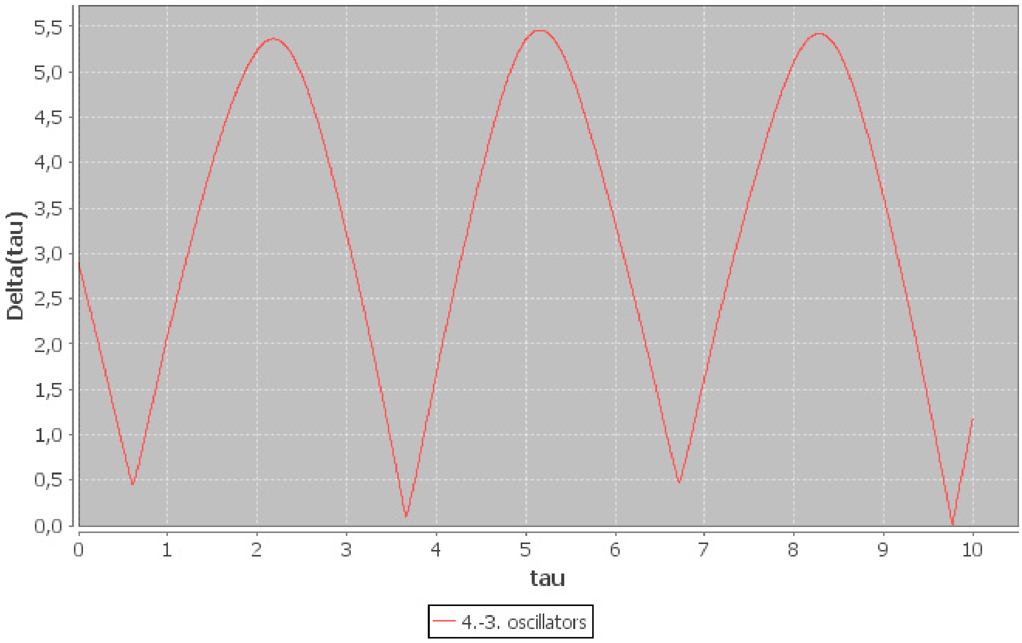

Figure 4.1.9 *Function* Δ, $j_1 = 3, j_2 = 4,$, $\varepsilon = 0.4$

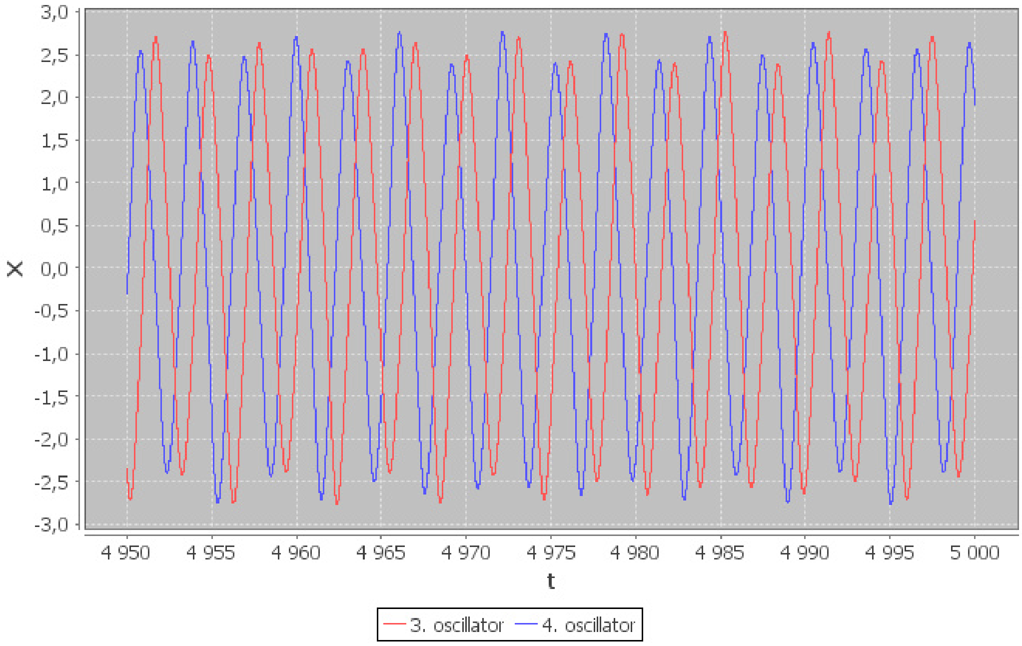

Figure 4.1.10 *Positions of oscillators 3. and 4. in time,* $\varepsilon = 0.4$.

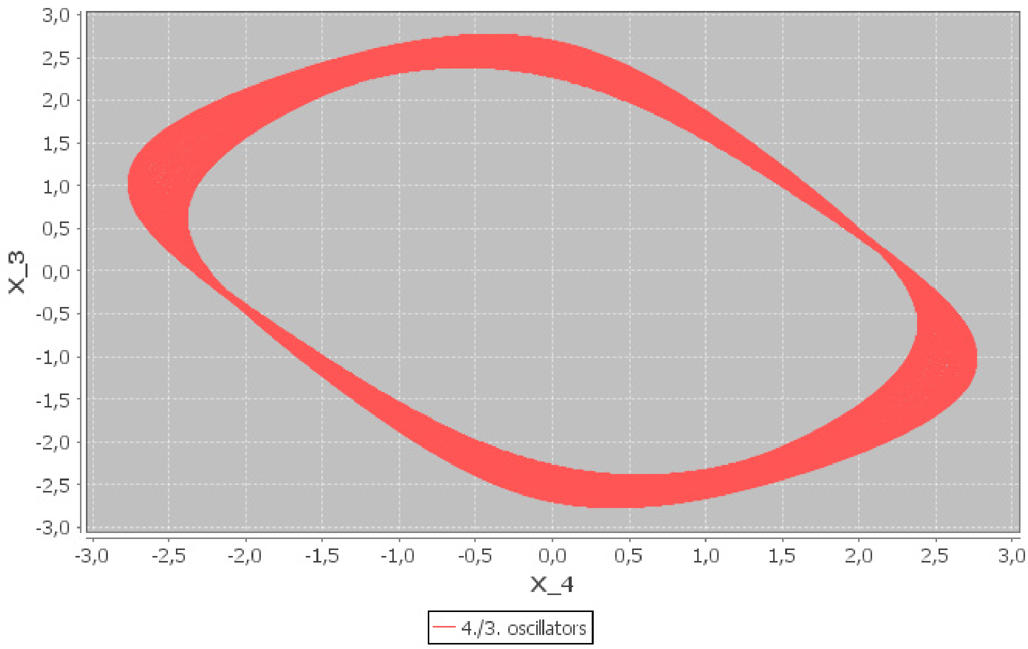

Figure 4.1.11 *Position of 3. oscillator versus position of 4. oscillator in time,* $\varepsilon = 0.4$.

From figure (4.1.10) it is hard to conclude that the oscillators are synchronized, but that suggests (4.1.11).
In addition, the figure (4.1.9) also confirms it. The local minimas of function Δ on interval $(0, 10)$ are in $\tau_1 = 0.61, \tau_2 = 3.665, \tau_3 = 6.715, \tau_4 = 9.77$ for which we have respectively $\Delta(\tau_1) = 0.4454, \Delta(\tau_2) = 0.094, \Delta(\tau_3) = 0.4638, \Delta(\tau_4) = 0.0145$. Putting the smallest obtained value $\delta := \Delta(\tau_4) = 0.0145$ the definition (4.1.2) is satisfied, because this value is approximately $0.52\% x^A_{4,1}$. So it can be concluded that oscillators are synchronized. It is possible, however, that for $\tau > 10$ is even better accuracy.
Results obtained above remain valid also for the rest of the pairs of coupled oscillators.
Beneath are charts after reducing the lag. For oscillator 3. we put time variable $t + 9.77$ and take its position with the opposite sign, and the oscillator 4. remains unchanged.

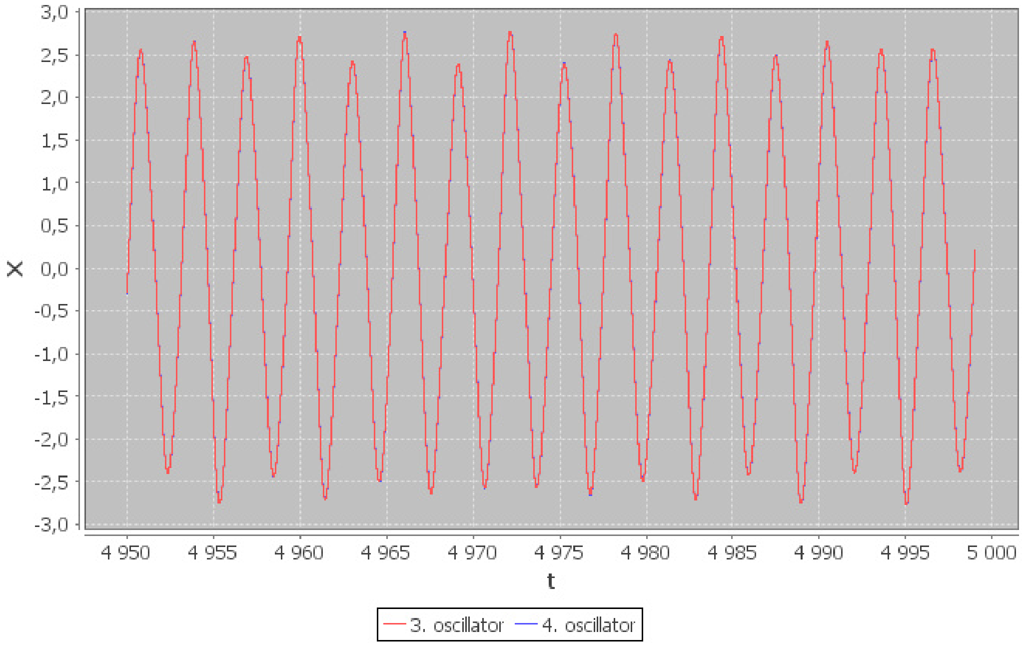

Figure 4.1.12 *Positions of oscillators 3. and 4.,* $\varepsilon = 0.4$.

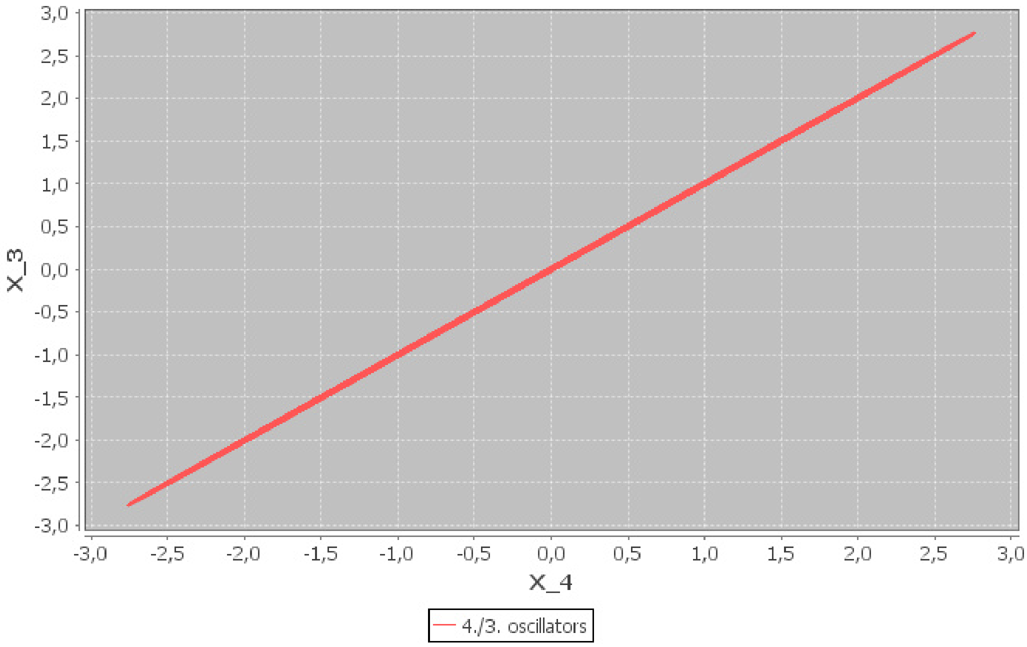

Figure 4.1.13 *Position of 3. versus position of 4. oscillator,* $\varepsilon = 0.4$.

Basing on the above, the answer to the main question of this paper is positive. There exist the value of the parameter ε such that the system (4.2) is synchronized. In the following, we have been searching for the lowest such value.

4.2 Moment of synchronization

We look for the smallest value of the parameter ε such for system (4.2) is synchronized. The concept of the phase of oscillator will be very useful here.

Definition 4.2.1 *The phase of the oscillator* $j \in \{1, \ldots, 10\}$ *is called function* $\phi_j : [0, \tilde{t}] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ *given by* $\phi_j(t) := \arctan(\frac{x_{j,2}(t)}{x_{j,1}(t)})$*. Where* $x_{j,1}(t) = 0$ *is assumed* $\phi_j(t) = \frac{\pi}{2}$ *or* $\phi_j(t) = -\frac{\pi}{2}$*, depending on the sign of* $x_{j,2}(t)$*. The case* $x_{j,1}(t) = x_{j,2}(t) = 0$ *is omitted, because it will not appear in this considerations. [2]*

For set t, above function defines the angle on the plane between the half-line $[0, \infty)$ and the segment which begins in point $(0, 0)$ and ends in point $(x_{j,1}(t), x_{j,2}(t))$.
If two oscillators are moving in phase space on the same path, or one of them on the path symmetrical across the point $(0, 0)$ to the path of the second (i.e. they are on the same attractor or on attractors symmetrical across the point $(0, 0)$), then the mean values of their phases $\tilde{\phi_j}$ are equal. This fact is very helpful, because when examining this mean values for a denoted value of ε it can be easily determined if it is the possible moment of the synchronization. But the resulted value of ε does not have to mean it, because this fact is only a necessary but not sufficient condition.
On beneath figures are the values of $\tilde{\phi_j}$. For every oscillator it is marked with individual colour.

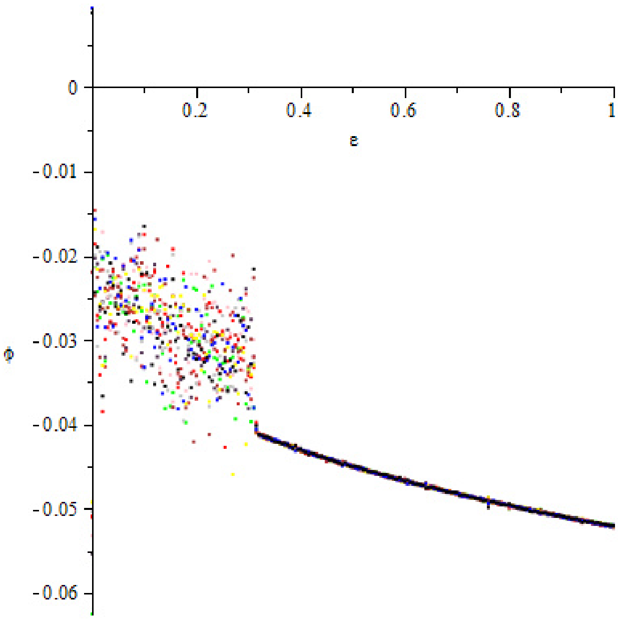

Figure 4.2.1 *Mean phase* $\tilde{\phi}_j$ *versus* ε*,* $\varepsilon \in [0, 1]$.

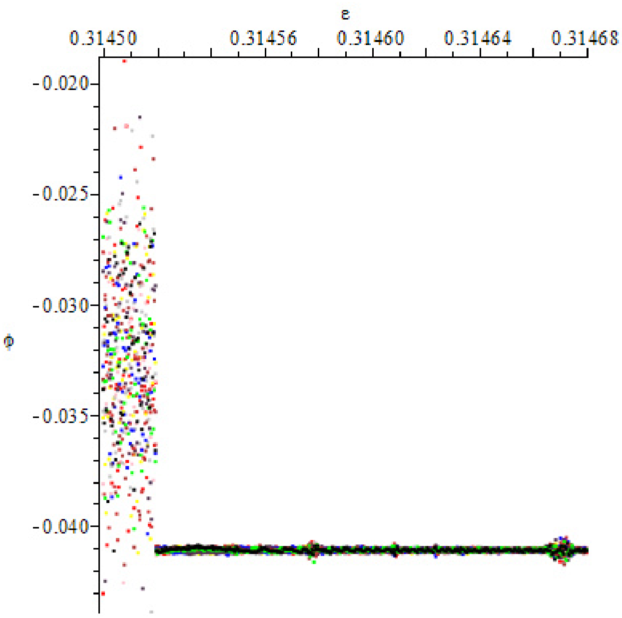

Figure 4.2.2 *Mean phase* $\tilde{\phi}_j$ *versus* ε*,* $\varepsilon \in [0.3145, 0.31468]$.

From (4.2.2) we can see that from some value of ε the points overlap. This moment may just mean synchronization. Indeed, putting $\varepsilon = 0.31452$:

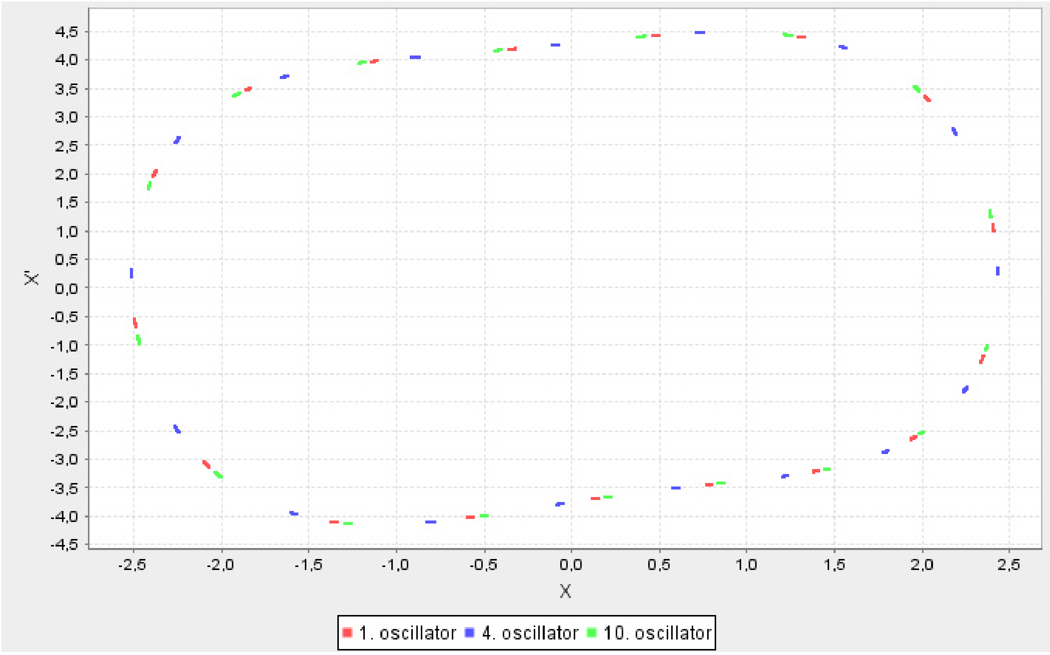

Figure 4.2.3 *Poincare maps for 1., 4. and 10. oscillators,* $\varepsilon = 0.31452$.

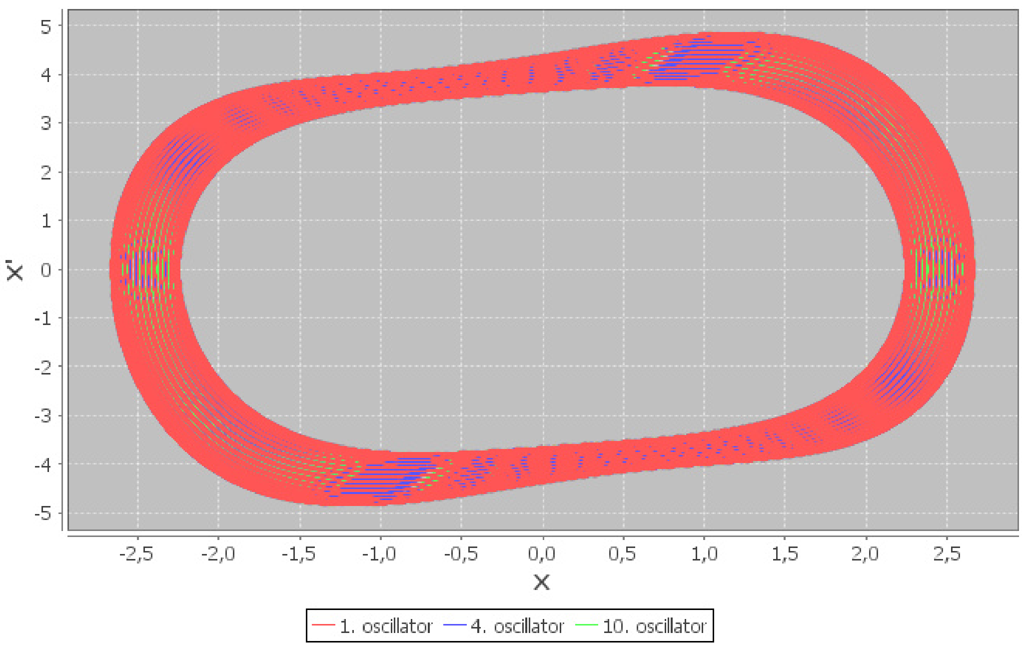

Figure 4.2.4 *Common attractor of 1., 4. and 10. oscillator,* $\varepsilon = 0.31452$.

Hence, the smallest value of ε such that system (4.2) is synchronized is $\varepsilon^* \approx 0.314605$. Moreover, on interval $\tau \in [0, 100]$ the best accuracy for each coupled pair of oscillators was obtained for $\tau = 22.84$. Then $\delta = 0.0102$ and its $0.38\% x^A_{j,1}$ for every j.

Increasing the value of ε more and more can be observed an interesting behavior of the system. It is presented on following figure.

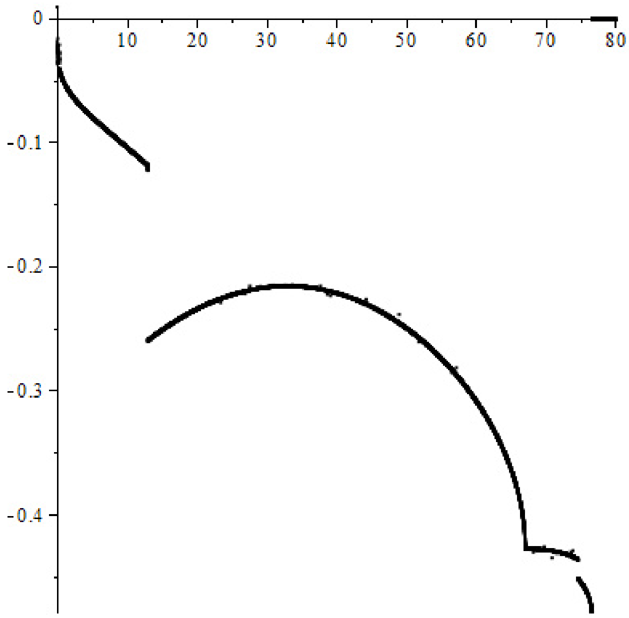

Figure 4.2.5 *Mean phase $\tilde{\phi}_1$ versus ε, $\varepsilon \in [0, 80]$.*

As on the figure (4.2.5), function has some points of discontinuity (we consider ε from the moment of synchronization). By studying the dynamics in these points we get the following.

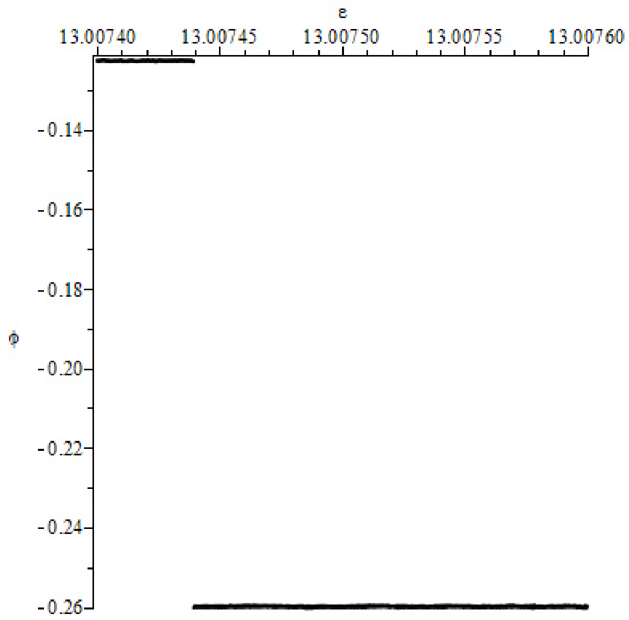

Figure 4.2.6 *Mean phase* $\tilde{\phi}_1$ *versus* ε*,* $\varepsilon \in [13.0074, 13.0076]$.

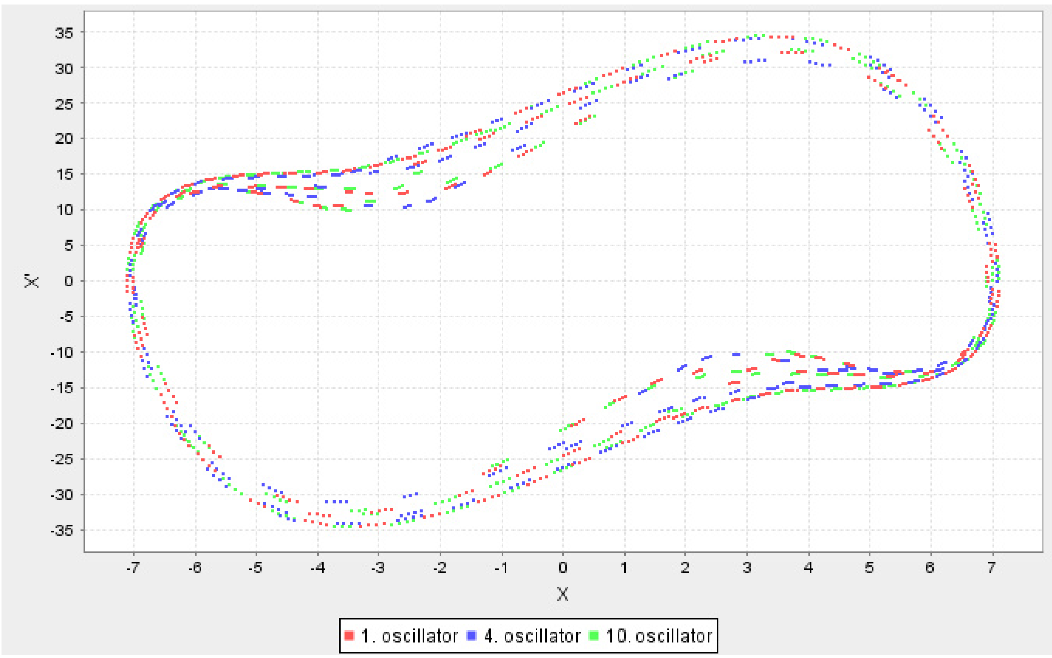

Figure 4.2.7 *Poincare maps for 1., 4. and 10. oscillators,* $\varepsilon \approx 13.0074$.

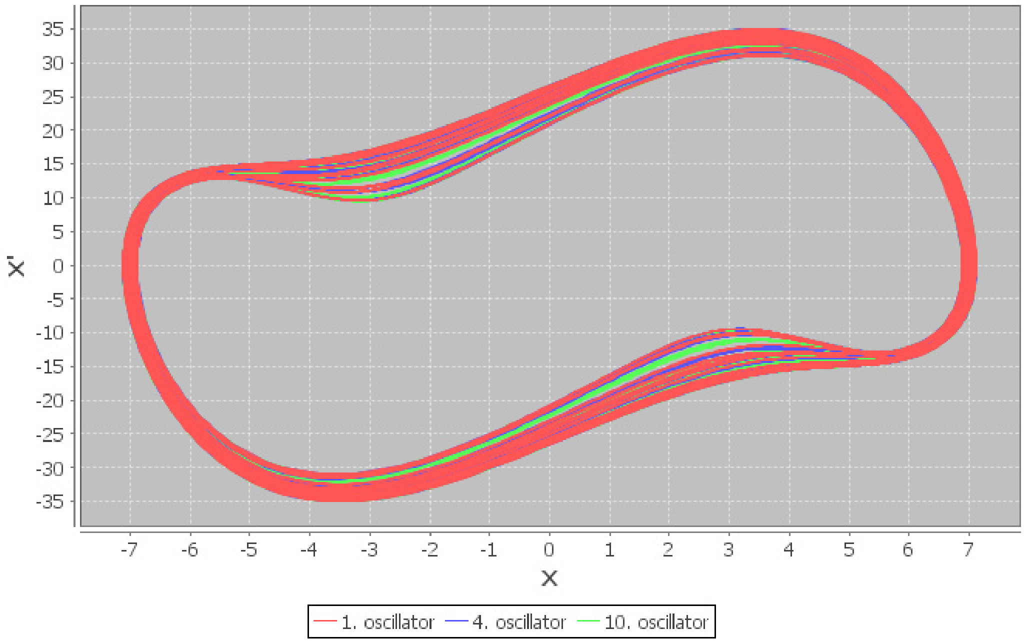

Figure 4.2.8 *Attractors of 1., 4. and 10. oscillator,* $\varepsilon \approx 13.0074$.

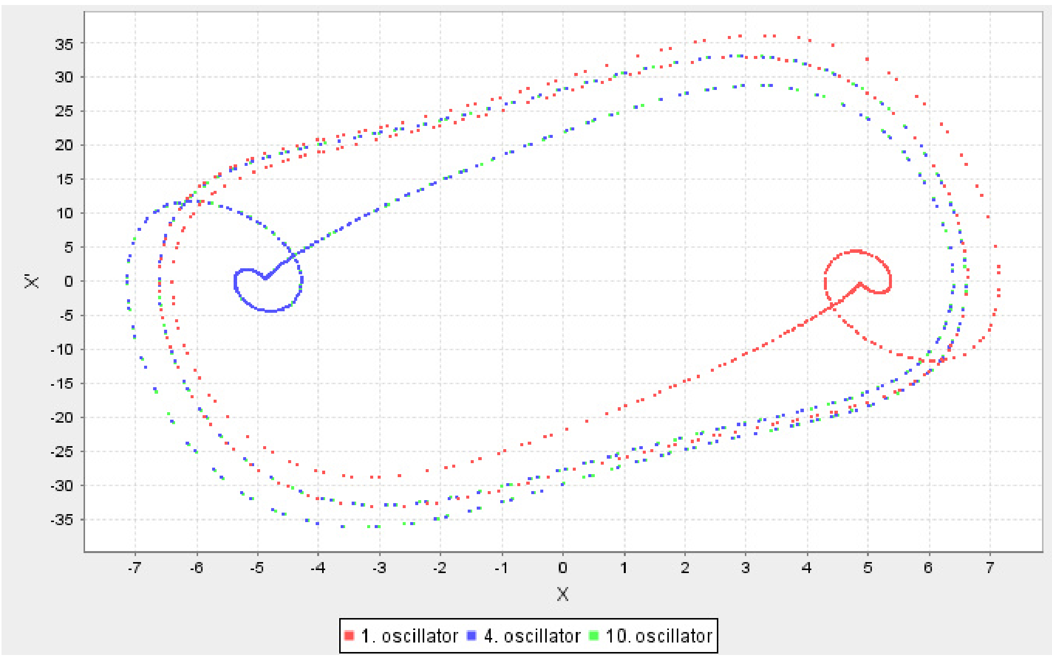

Figure 4.2.9 *Poincare maps for 1., 4. and 10. oscillators,* $\varepsilon \approx 13.00744$.

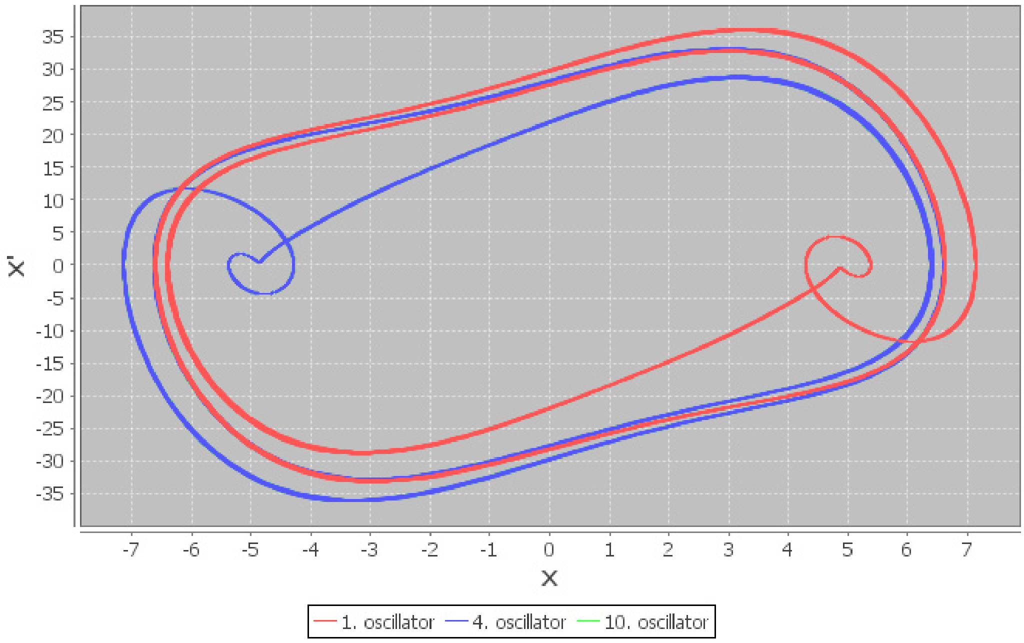

Figure 4.2.10 *Attractors of 1., 4. and 10. oscillator,* $\varepsilon \approx 13.00744$.

In this case, there appear two attractors, symmetrical across the point $(0, 0)$ on plane. Oscillators odd-numbered are on one attractor (red colour on (4.2.10)), and even-numbered on the second (blue colour). The moment of this change of the dynamics is $\varepsilon \approx 13.00744$.
Examining the synchronization for $\varepsilon = 13.00744$, on interval $\tau \in [0, 100]$ the best accuracy for each coupled pair of oscillators was obtained for $\tau = 42.485$. Hence, $\delta = 0.022$ which is $0.31\% x^A_{j,1}$ for every j.

The described behavior keeps until the one below.

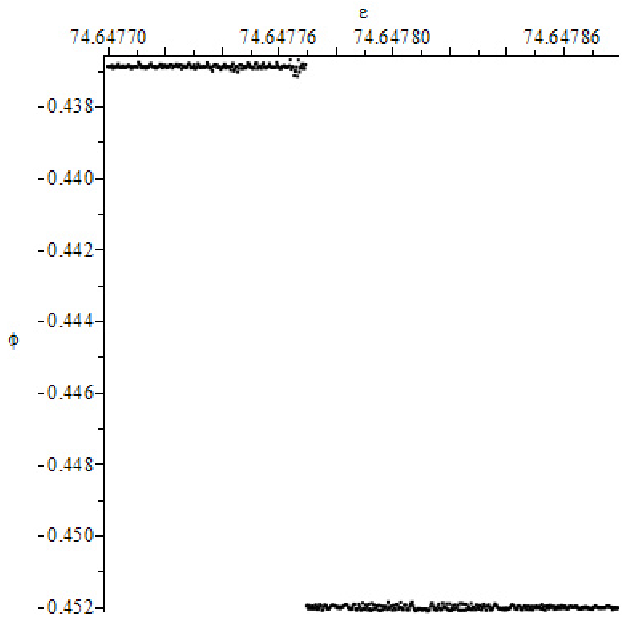

Figure 4.2.11 *Mean phase* $\tilde{\phi}_1$ *versus* ε, $\varepsilon \in [74.6477, 74.64788]$.

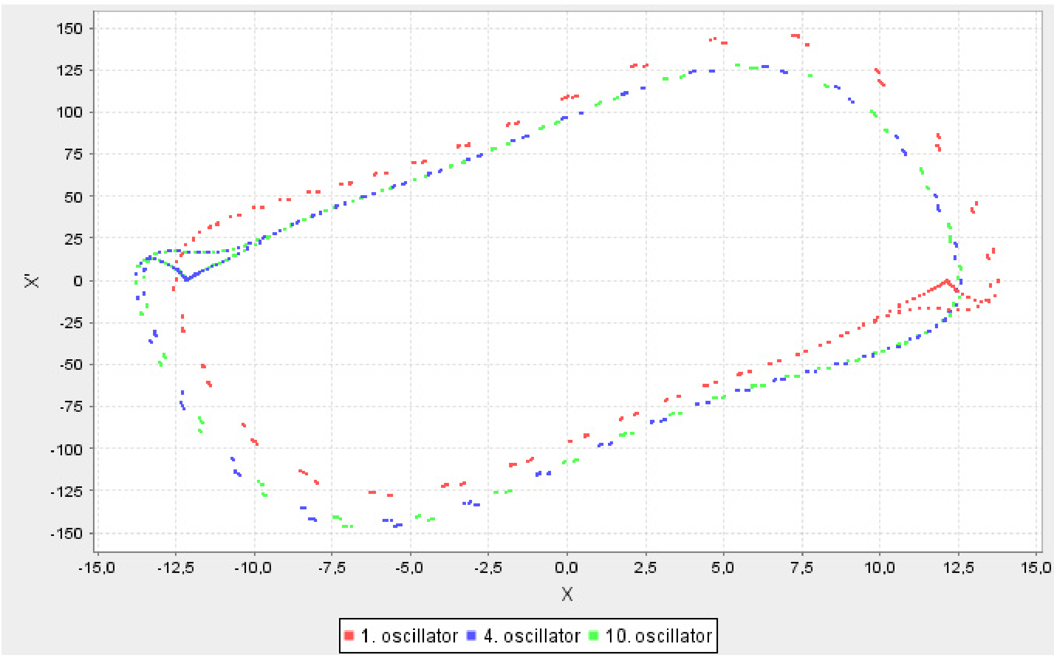

Figure 4.2.12 *Poincare maps for 1., 4. and 10. oscillators,* $\varepsilon \approx 74.6477$.

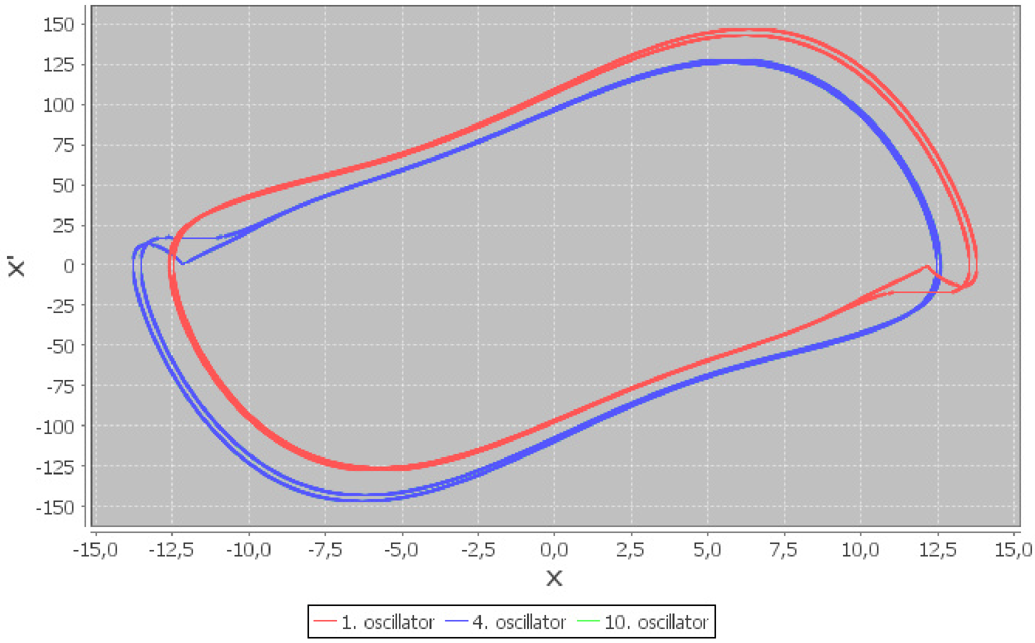

Figure 4.2.13 *Attractors of 1., 4. and 10. oscillator,* $\varepsilon \approx 74.6477$.

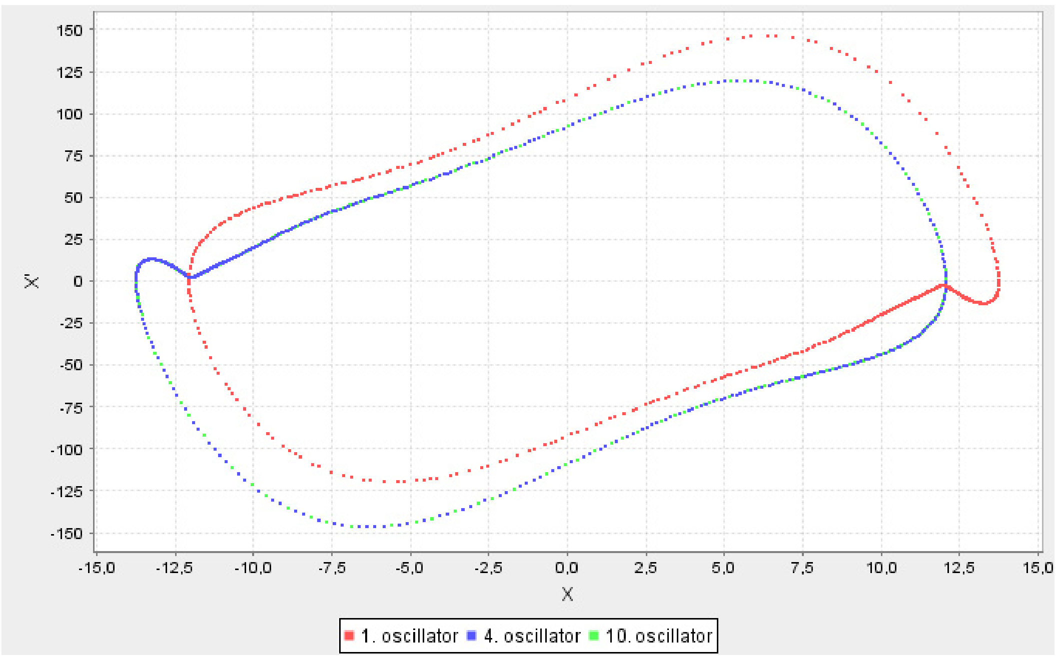

Figure 4.2.14 *Poincare maps for 1., 4. and 10. oscillators,* $\varepsilon \approx 74.64777$.

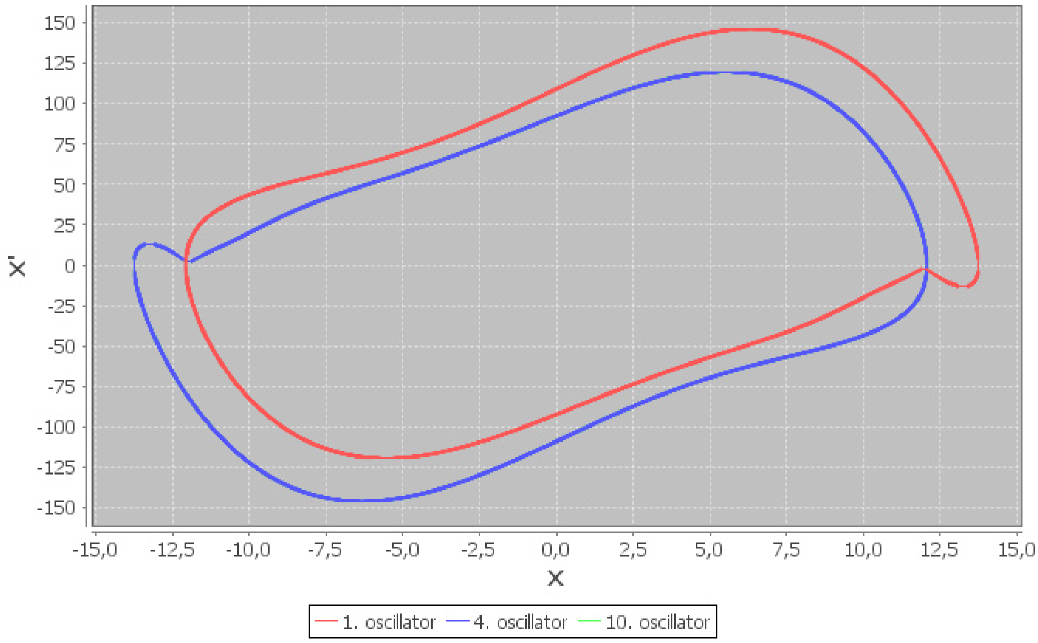

Figure 4.2.15 *Attractors of 1., 4. and 10. oscillator,* $\varepsilon \approx 74.64777$.

In this case, there still exists two attractors, symmetrical across the point $(0, 0)$ on plane, but the curve from figure (4.2.13) is doubling of the curve from (4.2.15). This phenomenom is called doubling of torus and can be found in [4].

Still oscillators odd-numbered are on one attractor (red colour on (4.2.15)), and even-numbered on the second (blue colour). The moment of this change in the dynamics is $\varepsilon \approx 74.64777$. Examining the synchronization for $\tau \in [0, 100]$ the best accuracy for each coupled pair of oscillators was obtained for $\tau = 87.36$. Hence, $\delta = 0.0215$ and its $0.16\% x^A_{j,1}$ for every j.

The described behavior keeps until the one below.

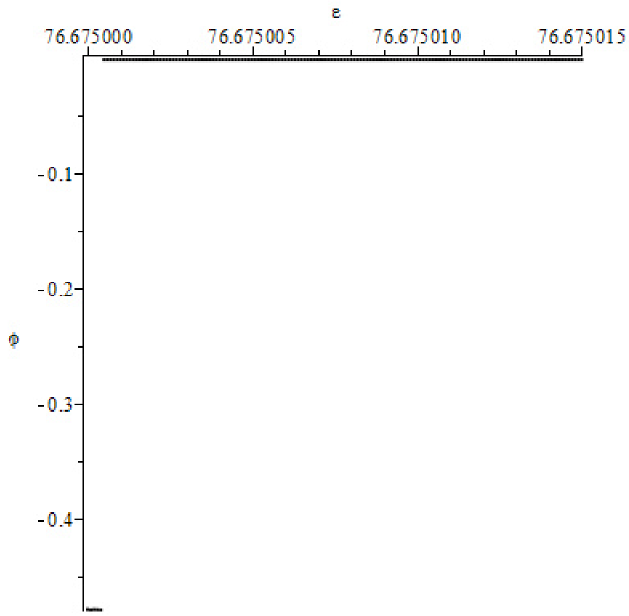

Figure 4.2.16 *Mean phase* $\tilde{\phi}_j$ *versus* ε, $\varepsilon \in [76.675, 76.675015]$.

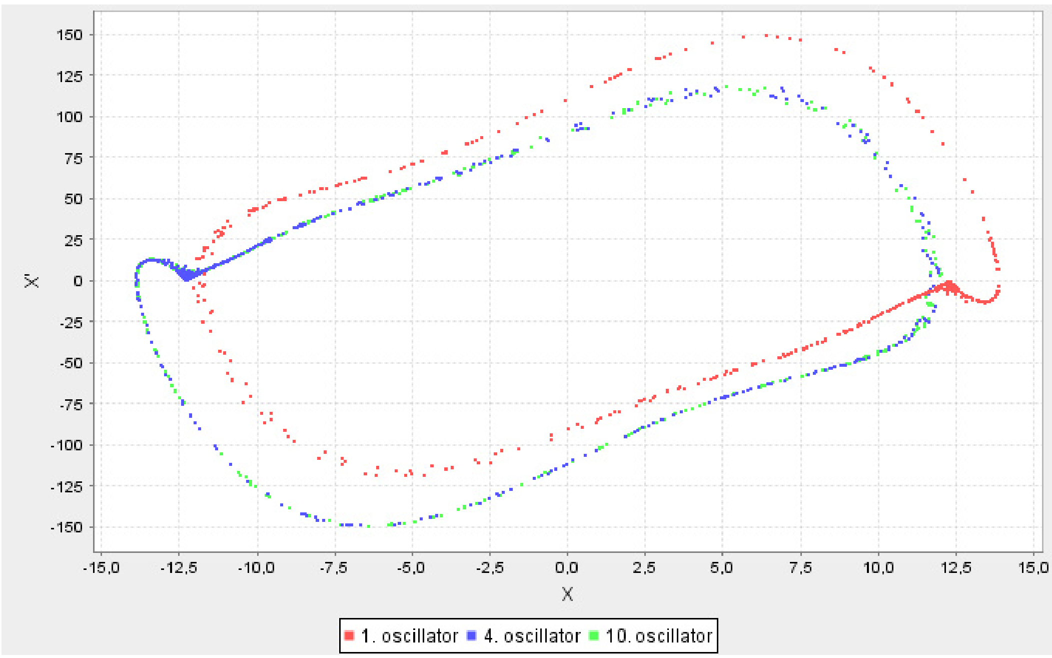

Figure 4.2.17 *Poincare maps for 1., 4. and 10. oscillators,* $\varepsilon \approx 76.675$.

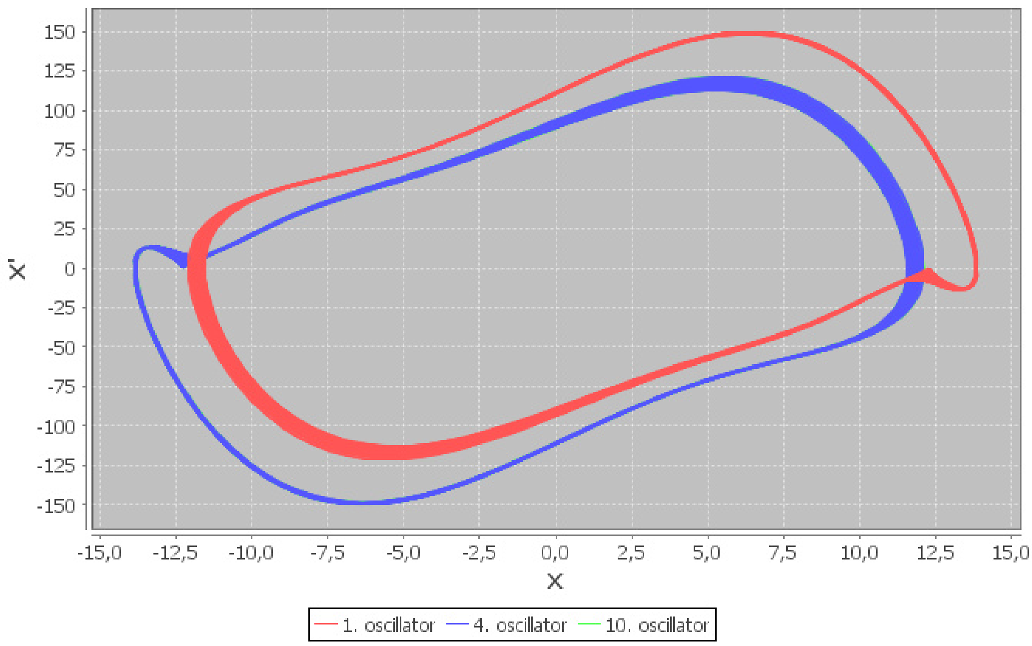

Figure 4.2.18 *Attractors of 1., 4. and 10. oscillator,* $\varepsilon \approx 76.675$.

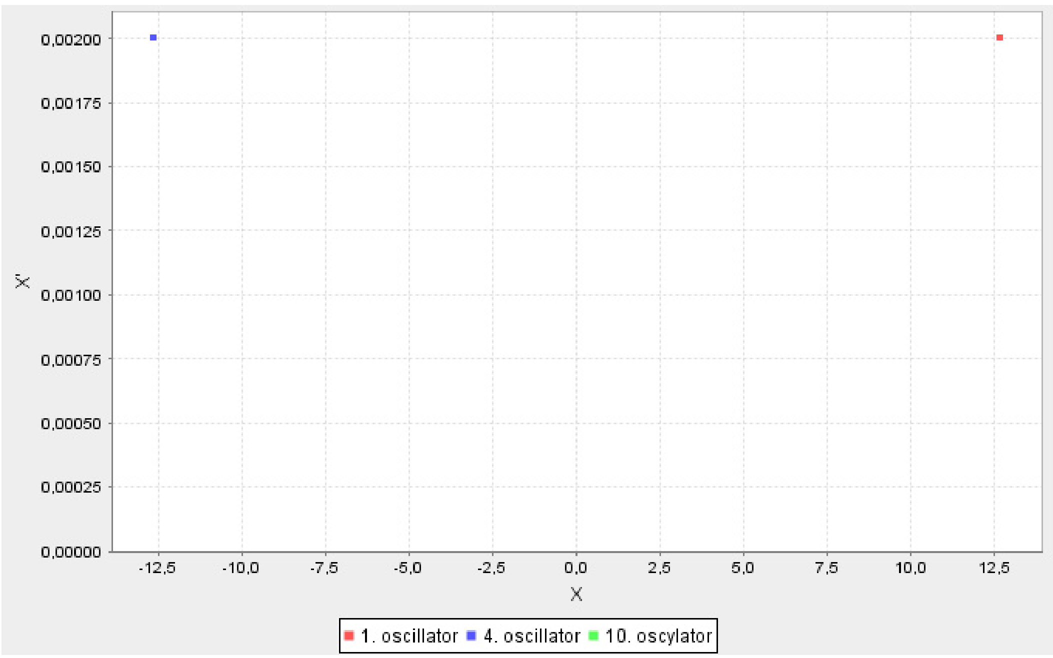

Figure 4.2.19 *Poincare maps for 1., 4. and 10. oscillators,* $\varepsilon \approx 76.67501$.

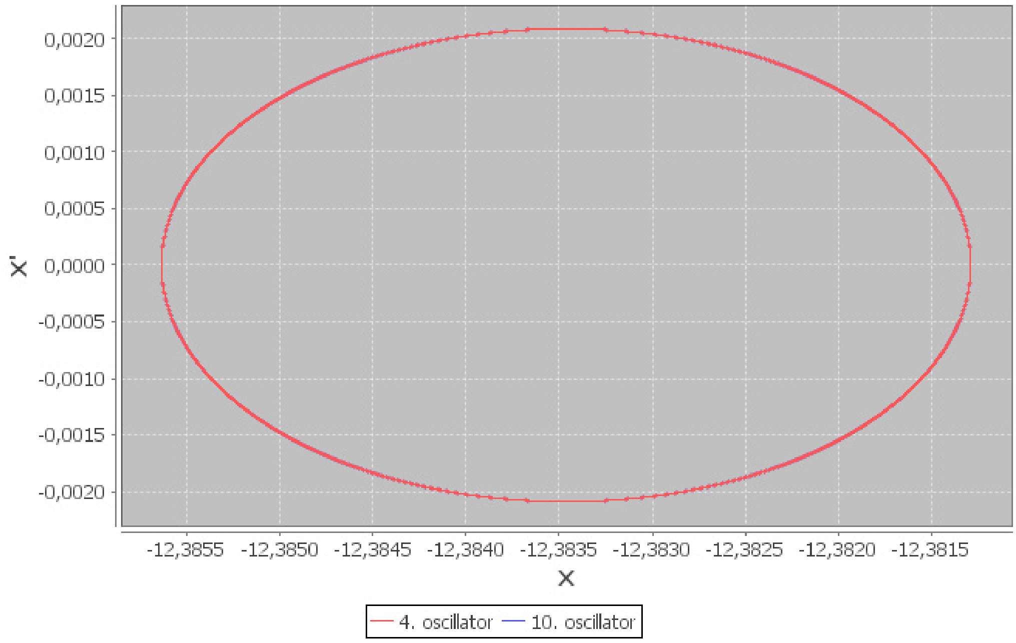

Figure 4.2.20 *Attractors of 4. and 10. oscillator,* $\varepsilon \approx 76.67501$.

Here is the case, which had to occur due to a constant increasing of ε. From a physical point of view, the coupling of oscillators as it is in the system (4.2) simulates a coupling of oscillators by spring, where the coupling parameter ε is its coefficient of elasticity. The bigger it is, the spring is getting stiffer. In this case, the strength of coupling (and thus the coefficient of elasticity) is so large that the spring begins to imitate rod. For this reason, the oscillators begins to vibrate periodically, which causes the external force expressed by the sine function in the equations of the problem (4.2). Solution for odd-numbered oscillators (red colour on (4.2.20)) is suitable multiple of function $\sin(0.962t)$, and for even-numbered oscillators (blue colour) the same function but with opposite sign. The moment of this change in the dynamics is $\varepsilon \approx 76.67501$. Because oscillators vibrate with the same frequency and amplitude, so after suitable lagging one of them of course we get the ideal synchronization, i.e. $\delta = 0$. Indeed, for $\tau = 3.265$ (nota bene, $\tau = 3.265 \approx \frac{2\pi}{2 \cdot 0.962}$), $\delta = 0.00000144$, which value can be considered as numerical zero.

From this case, no matter how much we increase the parameter, the dynamics can not change.

4.3 Disconnection of coupling

In this final subsection we study the dynamics of the synchronized system when the coupling element is disconnected. Very important here is the concept of rare attractor, which was introduced in the previous chapter.
To see the results better, we consider the problem on larger time horizon, putting $\tilde{t} = 10000$. Let us assume the following changes of the value of coupling parameter in the problem (4.2): when $t \in [0, 1000) \cup (5000, 10000]$ we put $\varepsilon = 6$ and for $t \in [1000, 5000]$, $\varepsilon = 0$.
For $t \in [0, 1000)$ oscillators are on one common attractor that resembles the shape of the one on figure (4.2.4). While $t = 1000$, the coupling is disconnected (system (4.1)). Below Poincare maps for oscillators after this operation are presented. The time $t \in [1000, 5000]$.

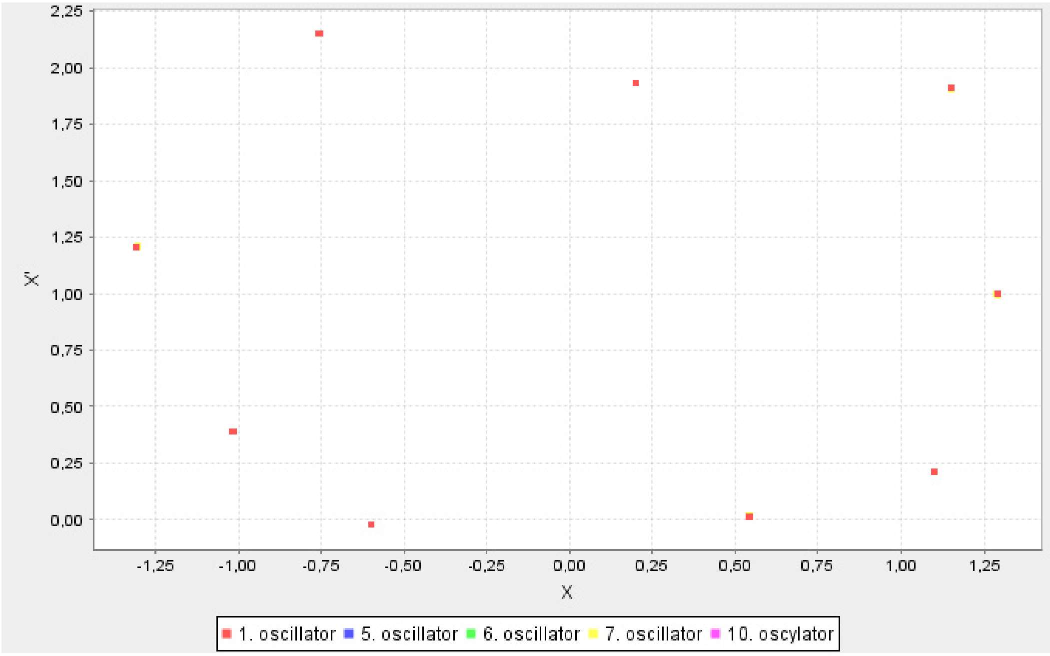

Figure 4.3.1 *Poincare maps for 1., 5., 6., 7. and 10. oscillator,* $\varepsilon = 0$.

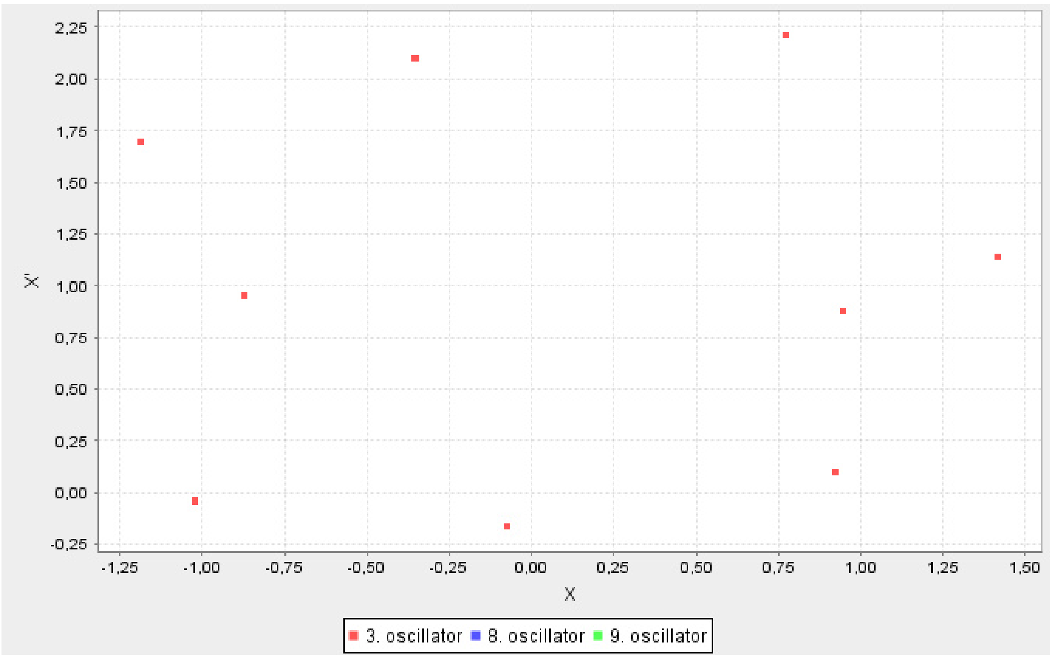

Figure 4.3.2 *Poincare maps for 3., 8. and 9. oscillator,* $\varepsilon = 0$.

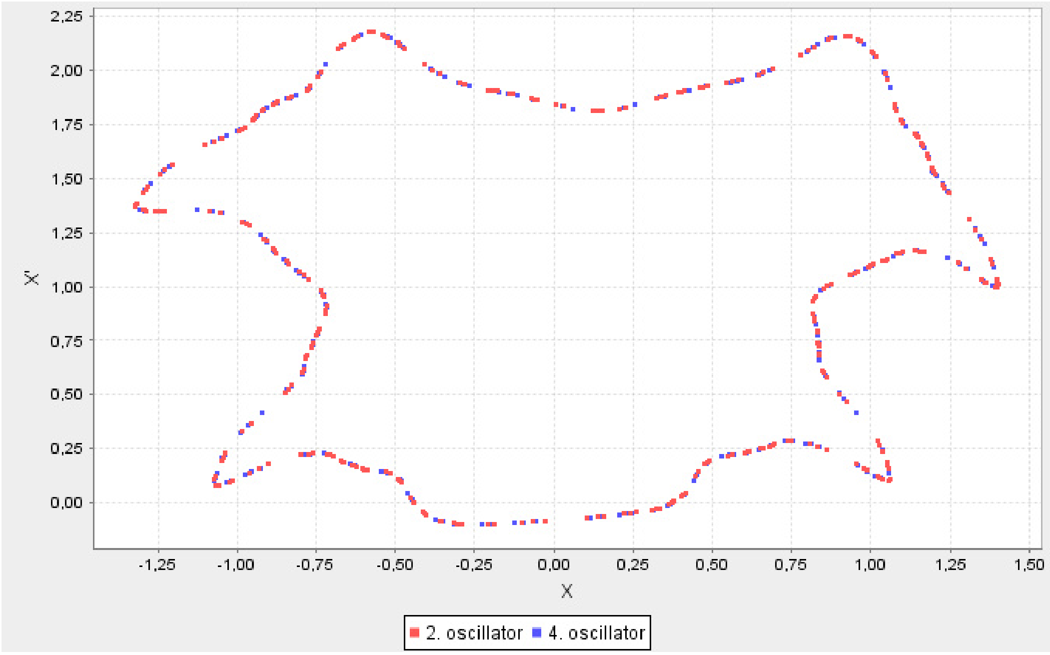

Figure 4.3.3 *Poincare maps for 2. and 4. oscillator, $\varepsilon = 0$.*

Oscillators 1., 5., 6., 7. and 10. are on $9\frac{2\pi}{0.962}$-periodic attractor, oscillators 3., 8. and 9. also on $9\frac{2\pi}{0.962}$-periodic, but different to the previous and 2. and 4. oscillators are on quasi-periodic attractor. Note that these attractors are the same attractors, from which previously started oscillators 1., 2. and 10. at the beginning of these chapter. The observed situation is closely connected with the theorem (3.2.3) from the previous chapter, which included the definition of rare attractor (definition (3.2.7)). Here we manipulate the set U of possible initial values. Denoting $U^1_{min} := \min\{x_{j,1}(1000) : j \in \{1, \ldots, 10\}\}, U^1_{max} := \max\{x_{j,1}(1000) : j \in \{1, \ldots, 10\}\},$ $U^2_{min} := \min\{x_{j,2}(1000) : j \in \{1, \ldots, 10\}\}, U^2_{max} := \max\{x_{j,2}(1000) : j \in \{1, \ldots, 10\}\}$ and assuming $U = [U^1_{min}, U^1_{max}] \times [U^2_{min}, U^2_{max}]$, the attractors other than that we have received are rare attractors, and thus the probability that the oscillator will be attracted by one of them is very small. Hence we obtain the result.

Another interesting phenomenon appears after reconnecting the coupling.

Continuing the discussion, for the time $t \in (5000, 10000]$, that is, when again $\varepsilon = 6$ we obtain the following Poincare maps.

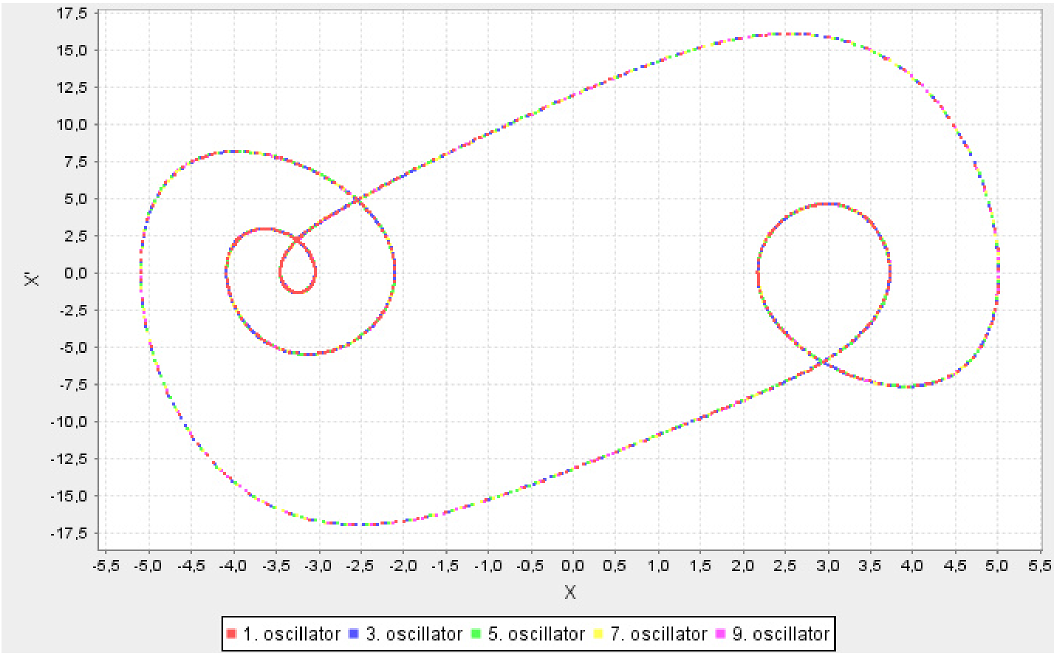

Figure 4.3.4 *Poincare maps for 1., 3., 4., 7. and 9. oscillator,* $\varepsilon = 6$.

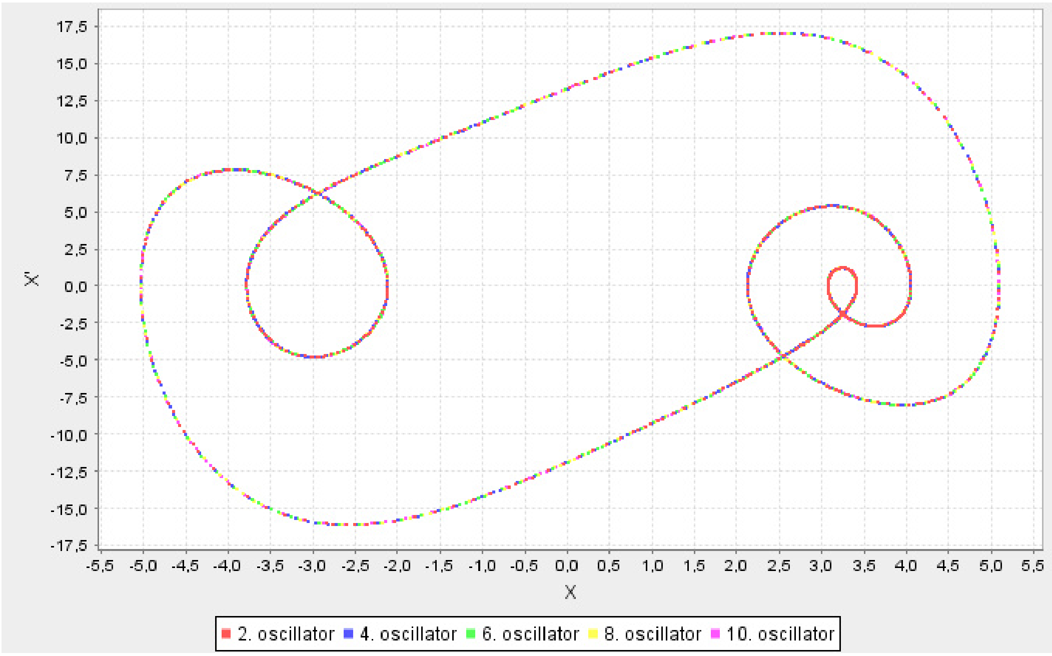

Figure 4.3.5 *Poincare maps for 2., 4., 6., 8. and 10. oscillator,* $\varepsilon = 6$.

It is phenomenon known from previous examples, when the odd-numbered oscillators are on one attractor and the even-numbered are on attractor that is symmetrical. However, here it takes place for more than two times lower value of the parameter ε. Previously, the moment for this change was the value $\varepsilon = 13.00754$, here its already for $\varepsilon = 6$ (possibly also for the lower value). The reasons for this behavior of the system should be seen on Poincare maps (4.3.1) - (4.3.3)), when the oscillators got attracted by non-rare attractors. When they again start from these attractors, the coupling strength $\varepsilon = 6$ is sufficient to appear the two symmetrical attractors.
The importance of the coupling strength can also be observed in other cases. When, for example, we carry out tests as above, but for $\varepsilon = 5$, it turns out that after reconnecting the coupling, we still obtain the common attractor. On the other hand, for $\varepsilon = 11$, in the end we get stiffness in the system and periodic oscillations. It is worth mentioning that this behavior may not be observed for all larger values of ε, because when at the beginning oscillators

starts not from common one attractor, but from two symmetrical, dynamics get different. For example, for $\varepsilon = 13.015$ after reconnecting there is no longer stiffness in the system, but there appear two symmetrical attractors.

Chapter 5

Conclusions

Van der Pol-Duffing oscillator is an issue very rich in terms of dynamics. The coexistence of multiple attractors, changes of their character under the influence of changes in factors etc. are only few examples to support this thesis. In this study, however, even though this problem is very complexity, by coupling together several of these oscillators we can observe standardized behavior, universal for each of them individually.

The lag synchronization is attained. For sufficient value of coupling parameter the oscillators are grouped on a common attractor in a surprising way. If we consider any coupled pair, we note that the function of the position versus time of one of the oscillators is a modification of such function of the other one. This graph is shifted in time for a fixed value (lag) and is taken with opposite sign. Surprisingly, this lag is universal for each such pair and it depends on both the coupling parameter and the initial conditions. Moreover, for the higher value of the coupling parameter, there appear two symmetrical attractors and oscillators are grouped on them equally (5 are on one and 5 are on the other one). It is fascinating that even such a significant change in the dynamics do not change the nature of the obtained synchronization.

It is also worth to mention, that we can track the changes in the behavior of common attractor (attractors) while increasing coupling parameter.

Even after breaking the common bonds, dynamics does not interfere in an unpredictable manner.

This results are only a starting point for further discussions. We focused here only on connecting identical systems. But other parameters may be set for each of the coupled oscillator, getting this way a new attractors (including chaotics) at the start. We might also consider some of them that are on the same attractor, but starting in its different points. A significant change should introduce an odd number of coupled oscillators, because in this case the situation when half of them converge to one attractor and the other half to the other is just impossible. This is of course due to the indivisibility of the even number by two. The behavior of the system with an odd number of coupled oscillators should not depend on the value of this number. It can be expected that instead of appearing more attractors system remains on one common and increasing the strength of the coupling parameter will only modify its shape. Even consideration of a simple system of the three oscillators should decide this hypothesis.

There are very many of such possibilities. We can expect that the studies about proposed variations of this problem, as well as others, will give researcher a whole of new, unusual observations.

Bibliography

[1] A. Chudzik, P. Perlikowski, A. Stefański, T. Kapitaniak, *Multistability and rare attractors in van der Pol - Duffing oscillator*, International Journal of Bifurcation and Chaos 21, 2011

[2] M. G. Rosenblum, A. S. Pikovsky, J. Kurths, *From Phase to Lag Synchronization in Coupled Chaotic Oscillators*, Physical Review Letters 78, 1997

[3] T. Kapitaniak, *Uncertainty in coupled chaotic systems: Locally intermingled basins of attraction*, Physical Review E 53, 1996

[4] G. O. Zhong, C.W. Wu, L.O. Chua, *Torus-Doubling Bifurcations in Four Mutually Coupled Chua's Circuits*, IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications vol. 45, no. 2, 1998

[5] A. Pikovsky, M. Rosenblum, J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences*, Cambridge University Press, 2001

[6] J. Awrejcewicz, *Matematyczne modelowanie systemów*, Wydawnictwa Naukowo-Techniczne, 2007

[7] W. Kołodziej, *Analiza matematyczna*, Wydawnictwo Naukowe PWN, 2010

[8] B. Przeradzki, *Teoria i praktyka równań różniczkowych zwyczajnych*, Wydawnictwo Uniwersytetu Łódzkiego, 2003

[9] A. Palczewski, *Równania różniczkowe zwyczajne*, Wydawnictwa Naukowo-Techniczne, 2004

[10] *http://im0.p.lodz.pl/ bprzeradzki/uk-dyn.pdf*

[11] *http://www.math.cornell.edu/ rand/randdocs/nlvibe53.pdf*